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New formulas and predictions for running fermion masses at higher scales in SM, 2HDM, and MSSM

C.R. Das[a], M.K. Parida[b]

Physics Department, North-Eastern Hill University, Shillong 793022, India

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Abstract. Including contributions of scale-dependent vacuum expectation values, we derive new analytic formulas and obtain substantially different numerical predictions for the running masses of quarks and charged leptons at higher scales in the SM, 2HDM and MSSM. These formulas exhibit significantly different behaviours with respect to their dependence on gauge and Yukawa couplings from those derived earlier. At one-loop level, the masses of the first two generations are found to be independent of the Yukawa couplings of the third generation in all three effective theories in the small mixing limit. Analytic formulas are also obtained for the running of $\tan\beta(\mu)$ in 2HDM and MSSM. Other numerical analyses include a study of the third generation masses at high scales as functions of the low-energy values of $\tan\beta$ and the SUSY scale $M_{\rm S} = M_Z - 10^4\,$GeV.

1 Introduction

One of the most attractive features of current investigations in gauge theories is the remarkable unification of the gauge couplings of the standard model (SM) at the SUSY GUT scale, $M_{\rm U} = 2 \times 10^{16}\,$GeV, when extrapolated through the minimal supersymmetric standard model (MSSM) [1]. Although the non-supersymmetric standard model (SM), or the two-Higgs doublet model (2HDM) do not answer the question of gauge hierarchy, unification of the gauge couplings is also possible at the corresponding GUT scales when they are embedded in non-SUSY theories like $SO(10)$, and the symmetry breaking takes place in two steps with left–right models as intermediate gauge symmetries [2]. Grand unification of gauge couplings of the SM in single-step breakings of GUTs has also been observed when the grand desert contains additional scalar degrees of freedom [3], and the minimal example is a $\xi(3,0,8)$ of SM contained in $\underline{75} \subset SU(5)$ or $\underline{210} \subset SO(10)$ with mass $M_\xi = 10^{11}$–$10^{13}\,$GeV [4]. Unification of gauge couplings in non-SUSY $SO(10)$ has been demonstrated with relatively large GUT threshold effects [5]. Yukawa coupling unification at the intermediate scale has also been observed in non-SUSY $SO(10)$ with 2HDM as the weak scale effective gauge theory [6]. Apart from the unity of forces at high scales, SM, 2HDM and MSSM have tremendous current importance as effective theories, as they emerge from a large class of fundamental theories.

Recent experimental evidences in favour of neutrino masses and mixings have triggered an outburst of models, many of which require running masses and mixings of quarks and charged leptons at high scales as inputs for obtaining predictions in the neutrino sector [7,8]. The running masses are not only essential at the weak scale, but they are also required at the intermediate and the GUT scales in order to testify theories based upon quark–lepton unification with different Yukawa textures and for providing a unified explanation of all fermion masses [9–13]. Quite recently, the extrapolation of running masses and couplings to high scales have been emphasised as an essential requirement for testing more fundamental theories [13].

In a recent paper one of us (M.K.P) and Purkayastha [14] have obtained new analytic formulas and numerical estimations for the fermion masses at higher scales in MSSM, including contributions of scale-dependent vacuum expectation values (VEVs), where the SUSY scale ($M_{\rm S}$) was assumed to be close to the weak scale ($M_{\rm S} \approx M_Z$). In this paper, we extend such investigations to SM, 2HDM and MSSM with the SUSY scale $M_{\rm S} \geq O$ (TeV).

It is also possible that in a different renormalisation scheme, similar to that formulated by Sirlin et al. [15], the VEVs themselves do not run when they are expressed in terms of physical parameters defined on the mass shell. This makes it possible to avoid separate running of the VEVs and Yukawa couplings, but to have just the fermion masses directly as running quantities. While it would be quite interesting to examine the consequences of such a scheme, the purpose of the present and recent works [14] is to address the outcome of the most frequently exploited renormalisation scheme where the Yukawa couplings and the VEVs run separately [16–24].

[a] e-mail: crdas@email.com
[b] e-mail: mparida@dte.vsnl.net.in

This paper is organised in the following manner. In Sect. 2 we cite examples where running VEVs have been exploited by a number of authors and state relevant renormalisation group equations (RGEs). In Sect. 3 we derive analytic formulas. In Sect. 4 we show how the formulas derived earlier for MSSM are modified when $M_{\mathrm{S}} \gg M_Z$. Section 5 gives a comparison of our formulas with other ones derived earlier in all the three gauge theories. Numerical predictions at higher scales are reported in Sect. 6. A summary and conclusions are stated in Sect. 7.

2 RGEs for couplings and vacuum expectation values

After the pioneering discovery of b–τ unification at the non-SUSY $SU(5)$ GUT scale [25], a number of theoretical investigations have been made to examine the behaviour of Yukawa couplings and running masses at higher scales. Following the frequently exploited renormalisation scheme [16–24] where the Yukawa couplings and the VEVs run separately, the running Dirac mass of the fermion "a" is defined by

$$M_a(\mu) = Y_a(\mu) v_a(\mu). \tag{2.1}$$

Then the running of $M_a(\mu)$ is governed both by the RGE of $Y_a(\mu)$ and $v_a(\mu)$. To cite some examples: Grimus [22] has derived approximate analytic formulas in SM for all values of μ extending up to the non-SUSY $SU(5)$ GUT scale utilising the corresponding scale-dependent VEV. In the discovery of fixed point Yukawa couplings, Pendleton and Ross [23] have exploited the RGE of the SM Higgs VEV to derive the RGEs of the running masses from $\mu = M_W - M_{\mathrm{GUT}}$. Anomalous dimensions occurring in the RGEs of respective VEVs have been explicitly derived and stated up to two loops by Arason et al. [16,17] and by Castano, Pirad and Ramond [18] for SM and MSSM. While investigating renormalisation of the neutrino mass operator, Babu, Leung and Pantaleone [24] have derived the RGE for $\tan\beta(\mu)$ in a class of 2HDM as a consequence of running VEVs in the model. More recently, Balzeleit et al. [20] have utilised the RGE of the VEV in SM to determine running masses for $\mu = M_W - 10^{10}$ GeV. Cvetic, Hwang and Kim [21] have derived RGEs for the VEVs in 2HDM and utilised them to obtain running quark–lepton masses at high scales and also investigate the suppression of a flavour changing neutral current in the model. Most recently the RGEs of running VEVs have been utilised by one of us (M.K.P.) and Purkayastha [14] who have obtained new analytic formulas and numerical estimations of the fermion masses at higher scales taking the SUSY scale $M_{\mathrm{S}} \approx M_Z$.

We consider only the class of 2HDM where Φ_u gives masses to up quarks and Φ_d to down quarks and charged leptons. For the sake of simplicity we ignore the neutrino mass in the present paper; this will be addressed separately. Our definitions and conventions for the Yukawa couplings and masses are governed by the following Yukawa Lagrangian (superpotential) in SM or 2HDM (MSSM) and the corresponding VEVs of Higgs scalars:

SM

$$\begin{aligned} \mathcal{L}_Y &= \overline{Q}_{\mathrm{L}} Y_U \tilde{\Phi} U_{\mathrm{R}} + \overline{Q}_{\mathrm{L}} Y_D \Phi D_{\mathrm{R}} + \bar{l}_{\mathrm{L}} Y_E \Phi E_{\mathrm{R}} + \text{h.c.}, \\ \langle \Phi^0(\mu) \rangle &= v(\mu). \end{aligned} \tag{2.2}$$

2HDM, MSSM

$$\begin{aligned} \mathcal{L}_Y &= \overline{Q}_{\mathrm{L}} Y_U \Phi_u U_{\mathrm{R}} + \overline{Q}_{\mathrm{L}} Y_D \Phi_d D_{\mathrm{R}} \\ &\quad + \bar{l}_{\mathrm{L}} Y_E \Phi_d E_{\mathrm{R}} + \text{h.c.}, \\ \langle \Phi_u^0(\mu) \rangle &= v_u(\mu) = v(\mu)\sin\beta(\mu), \\ \langle \Phi_d^0(\mu) \rangle &= v_d(\mu) = v(\mu)\cos\beta(\mu), \\ v^2(\mu) &= v_u^2(\mu) + v_d^2(\mu), \\ \tan\beta(\mu) &= v_u(\mu)/v_d(\mu). \end{aligned} \tag{2.3}$$

The relevant RGEs for the Yukawa matrices at one-loop level for the three effective theories are expressed as [16–19,26–28]

$$\begin{aligned} 16\pi^2 \frac{\mathrm{d}Y_U}{\mathrm{d}t} &= \Big[\mathrm{Tr}\left(3Y_U Y_U^\dagger + 3aY_D Y_D^\dagger + aY_E Y_E^\dagger\right) \\ &\quad + \frac{3}{2}\left(bY_U Y_U^\dagger + cY_D Y_D^\dagger\right) - \sum_i C_i^{(u)} g_i^2\Big] Y_U, \\ 16\pi^2 \frac{\mathrm{d}Y_D}{\mathrm{d}t} &= \Big[\mathrm{Tr}\left(3aY_U Y_U^\dagger + 3Y_D Y_D^\dagger + Y_E Y_E^\dagger\right) \\ &\quad + \frac{3}{2}\left(bY_D Y_D^\dagger + cY_U Y_U^\dagger\right) - \sum_i C_i^{(d)} g_i^2\Big] Y_D, \\ 16\pi^2 \frac{\mathrm{d}Y_E}{\mathrm{d}t} &= \Big[\mathrm{Tr}\left(3aY_U Y_U^\dagger + 3Y_D Y_D^\dagger + Y_E Y_E^\dagger\right) \\ &\quad + \frac{3}{2} bY_E Y_E^\dagger - \sum_i C_i^{(e)} g_i^2\Big] Y_E. \end{aligned} \tag{2.4}$$

The RGEs for the VEV in the SM have been derived up to two loops from wave function renormalisation of the scalar field [16,17,19,20,22,23] and the one-loop equation is

$$16\pi^2 \frac{\mathrm{d}v}{\mathrm{d}t} = \left[\sum_i C_i^v g_i^2 - \mathrm{Tr}\left(3Y_U Y_U^\dagger + 3Y_D Y_D^\dagger + Y_E Y_E^\dagger\right)\right] v, \tag{2.5}$$

where $t = \ln\mu$.

The RGEs for $v_a (a = u, d)$ in the 2HDM up to one loop and in MSSM up to two loops have been derived in [16–19,21]. The one-loop equations in both theories are

$$\begin{aligned} 16\pi^2 \frac{\mathrm{d}v_u}{\mathrm{d}t} &= \left[\sum_i C_i^v g_i^2 - \mathrm{Tr}\left(3Y_U Y_U^\dagger\right)\right] v_u, \\ 16\pi^2 \frac{\mathrm{d}v_d}{\mathrm{d}t} &= \left[\sum_i C_i^v g_i^2 - \mathrm{Tr}\left(3Y_D Y_D^\dagger + Y_E Y_E^\dagger\right)\right] v_d; \end{aligned} \tag{2.6}$$

whereas charged lepton Yukawa contributions were ignored in the R.H.S. of (2.6) for the 2HDM in [21], we have included them. The gauge couplings in the three models obey the well-known one-loop RGEs:

$$16\pi^2 \frac{\mathrm{d}g_i}{\mathrm{d}t} = b_i g_i^3. \tag{2.7}$$

Two-loop contributions have been derived by a number of authors [16–19,22–28]. The coefficients appearing in the R.H.S. of (2.4)–(2.7) are defined in the three different cases:

SM, 2HDM

$$\begin{aligned} C_i^u &= \left(\frac{17}{20}, \frac{9}{4}, 8\right), \\ C_i^d &= \left(\frac{1}{4}, \frac{9}{4}, 8\right), \\ C_i^e &= \left(\frac{9}{4}, \frac{9}{4}, 0\right), \\ C_i^v &= \left(\frac{9}{20}, \frac{9}{4}, 0\right). \end{aligned} \tag{2.8}$$

MSSM

$$\begin{aligned} C_i^u &= \left(\frac{13}{15}, 3, \frac{16}{3}\right), \\ C_i^d &= \left(\frac{7}{15}, 3, \frac{16}{3}\right), \\ C_i^e &= \left(\frac{9}{5}, 3, 0\right), \\ C_i^v &= \left(\frac{3}{20}, \frac{3}{4}, 0\right). \end{aligned} \tag{2.9}$$

SM

$$\begin{aligned} b_i &= \left(\frac{41}{10}, -\frac{19}{6}, -7\right), \\ (a, b, c) &= (1, 1, -1). \end{aligned} \tag{2.10}$$

2HDM

$$\begin{aligned} b_i &= \left(\frac{21}{5}, -3, -7\right), \\ (a, b, c) &= \left(0, 1, \frac{1}{3}\right). \end{aligned} \tag{2.11}$$

MSSM

$$\begin{aligned} b_i &= \left(\frac{33}{5}, 1, -3\right), \\ (a, b, c) &= \left(0, 2, \frac{2}{3}\right). \end{aligned} \tag{2.12}$$

For the sake of simplicity we have neglected the Yukawa interactions of the neutrinos. Assuming that the right-handed neutrinos are massive ($M_N > 10^{13}$ GeV) our formulas are valid below M_N to a very good approximation even if such interactions are included.

3 RGEs and analytic formulas for running masses

Using the definition (2.1) and (2.4)–(2.12), we obtain the RGEs for the mass matrices in the broken phases of SM, 2HDM, or MSSM in the following form:

$$\begin{aligned} 16\pi^2 \frac{\mathrm{d}M_U}{\mathrm{d}t} &= \left(-\sum_i C_i g_i^2 + \tilde{a} Y_U Y_U^\dagger + \tilde{b} Y_D Y_D^\dagger\right) M_U, \\ 16\pi^2 \frac{\mathrm{d}M_D}{\mathrm{d}t} &= \left(-\sum_i C_i' g_i^2 + \tilde{b} Y_U Y_U^\dagger + \tilde{a} Y_D Y_D^\dagger\right) M_D, \\ 16\pi^2 \frac{\mathrm{d}M_E}{\mathrm{d}t} &= \left(-\sum_i C_i'' g_i^2 + \tilde{c} Y_E Y_E^\dagger\right) M_E, \end{aligned} \tag{3.1}$$

where the coefficients in the R.H.S. are defined for the three cases:

SM, 2HDM

$$\begin{aligned} C_i &= \left(\frac{2}{5}, 0, 8\right), \\ C_i' &= \left(-\frac{1}{5}, 0, 8\right), \\ C_i'' &= \left(\frac{9}{5}, 0, 0\right). \end{aligned} \tag{3.2}$$

MSSM

$$\begin{aligned} C_i &= \left(\frac{43}{60}, \frac{9}{4}, \frac{16}{3}\right), \\ C_i' &= \left(\frac{19}{60}, \frac{9}{4}, \frac{16}{3}\right), \\ C_i'' &= \left(\frac{33}{20}, \frac{9}{4}, 0\right), \\ \left(\tilde{a}, \tilde{b}, \tilde{c}\right) &= (3, 1, 3). \end{aligned} \tag{3.3}$$

SM

$$\left(\tilde{a}, \tilde{b}, \tilde{c}\right) = \left(\frac{3}{2}, -\frac{3}{2}, \frac{3}{2}\right). \tag{3.4}$$

2HDM

$$\left(\tilde{a}, \tilde{b}, \tilde{c}\right) = \left(\frac{3}{2}, \frac{1}{2}, \frac{3}{2}\right). \tag{3.5}$$

Defining the diagonal mass matrices $\hat{M}_F$, the diagonal Yukawa matrices ($\hat{Y}_F$) and the CKM matrix (V) through biunitary transformations L_F and R_F on the left(right)-handed fermion $F_\mathrm{L}(F_\mathrm{R})$ with $F = U, D, E$,

$$\begin{aligned} \hat{M}_F &= L_F^\dagger M_F R_F, \\ \hat{Y}_F &= L_F^\dagger Y_F R_F, \\ \hat{M}_F^2 &= L_F^\dagger M_F M_F^\dagger L_F, \\ \hat{Y}_F^2 &= L_F^\dagger Y_F Y_F^\dagger L_F, \\ V &= L_U^\dagger L_D, \end{aligned} \tag{3.6}$$

and following the procedures outlined in [14,29], we obtain

$$\begin{aligned} \frac{\mathrm{d}\hat{M}_U^2}{\mathrm{d}t} &= \left[\hat{M}_U^2, L_U^\dagger \dot{L}_U\right] + \frac{1}{16\pi^2}\left[-2\sum_i C_i g_i^2 \hat{M}_U^2\right. \\ &\quad \left. + 2\tilde{a}\hat{Y}_U^2 \hat{M}_U^2 + \tilde{b}\left(V\hat{Y}_D^2 V^\dagger \hat{M}_U^2 + \hat{M}_U^2 V \hat{Y}_D^2 V^\dagger\right)\right], \end{aligned}$$

$$\frac{\mathrm{d}\hat{M}_D^2}{\mathrm{d}t} = \left[\hat{M}_D^2, L_D^\dagger \dot{L}_D\right] + \frac{1}{16\pi^2}\left[-2\sum_i C_i' g_i^2 \hat{M}_D^2 + 2\tilde{a}\hat{Y}_D^2\hat{M}_D^2 + \tilde{b}\left(V^\dagger \hat{Y}_U^2 V \hat{M}_D^2 + \hat{M}_D^2 V^\dagger \hat{Y}_U^2 V\right)\right],$$
$$\frac{\mathrm{d}\hat{M}_E^2}{\mathrm{d}t} = \left[\hat{M}_E^2, L_E^\dagger \dot{L}_E\right] + \frac{1}{16\pi^2}\left[-2\sum_i C_i'' g_i^2 \hat{M}_E^2 + 2\tilde{c}\hat{Y}_E^2\hat{M}_E^2\right], \tag{3.7}$$

where $\dot{L}_F = \mathrm{d}L_F/\mathrm{d}t$.

We point out that in the corresponding RGEs for Yukawa couplings given by (2.13) in [29], the terms $-2\sum_i C_i^u g_i^2 \hat{Y}_U^2/\left(16\pi^2\right)$, $-2\sum_i C_i^d g_i^2 \hat{Y}_D^2/\left(16\pi^2\right)$ and $-2\sum_i C_i^e g_i^2 \hat{Y}_E^2/\left(16\pi^2\right)$ are missing from the R.H.S.

The diagonal elements of $L_F^\dagger \dot{L}_F$ $(F = U, D, E)$ are made to vanish in the usual manner by diagonal phase multiplication. The non-diagonal elements of both sides of (3.7) give the same RGEs for the CKM matrix elements as before, which on integration yields [29,30]

$$|V_{\alpha\beta}(\mu)| = \begin{cases} |V_{\alpha\beta}(m_t)| \exp\left(\frac{3}{2}c\left(I_t(\mu) + I_b(\mu)\right)\right), \\ \quad \alpha\beta = ub, cb, tb, ts, \\ |V_{\alpha\beta}(m_t)|, \\ \quad \text{otherwise.} \end{cases} \tag{3.8}$$

Taking diagonal elements of both sides of (3.7) and using dominance of the Yukawa couplings of the third generation over the first two, except the charm quark, we obtain RGEs for the mass eigenvalues of quarks and leptons:

$$\begin{aligned}
16\pi^2\frac{\mathrm{d}m_u}{\mathrm{d}t} &= \left[-\sum_i C_i g_i^2 + \tilde{b}y_b^2|V_{ub}|^2\right] m_u, \\
16\pi^2\frac{\mathrm{d}m_c}{\mathrm{d}t} &= \left[-\sum_i C_i g_i^2 + \tilde{a}y_c^2 + \tilde{b}y_b^2|V_{cb}|^2\right] m_c, \\
16\pi^2\frac{\mathrm{d}m_t}{\mathrm{d}t} &= \left[-\sum_i C_i g_i^2 + \tilde{a}y_t^2 + \tilde{b}y_b^2|V_{tb}|^2\right] m_t, \\
16\pi^2\frac{\mathrm{d}m_j}{\mathrm{d}t} &= \left[-\sum_i C_i' g_i^2 + \tilde{b}y_t^2|V_{tj}|^2\right] m_j, \quad j = d, s, \\
16\pi^2\frac{\mathrm{d}m_b}{\mathrm{d}t} &= \left[-\sum_i C_i' g_i^2 + \tilde{a}y_b^2 + \tilde{b}y_t^2|V_{tb}|^2\right] m_b, \\
16\pi^2\frac{\mathrm{d}m_j}{\mathrm{d}t} &= \left[-\sum_i C_i'' g_i^2\right] m_j, \quad j = e, \mu, \\
16\pi^2\frac{\mathrm{d}m_\tau}{\mathrm{d}t} &= \left[-\sum_i C_i'' g_i^2 + \tilde{c}y_\tau^2\right] m_\tau.
\end{aligned} \tag{3.9}$$

Integrating (3.9) and using the corresponding low-energy values, the new analytic formulas are obtained in the small mixing limit:

$$\begin{aligned}
m_u(\mu) &= m_u(1\,\mathrm{GeV})\eta_u^{-1}B_u^{-1}, \\
m_c(\mu) &= m_c(m_c)\eta_c^{-1}B_u^{-1}\exp(\tilde{a}I_c), \\
m_t(\mu) &= m_t(m_t)B_u^{-1}\exp(\tilde{a}I_t + \tilde{b}I_b), \\
m_i(\mu) &= m_i(1\,\mathrm{GeV})\eta_i^{-1}B_d^{-1}, \quad i = d, s, \\
m_b(\mu) &= m_b(m_b)\eta_b^{-1}B_d^{-1}\exp(\tilde{a}I_b + \tilde{b}I_t), \\
m_i(\mu) &= m_i(1\,\mathrm{GeV})\eta_i^{-1}B_e^{-1}, \quad i = e, \mu, \\
m_\tau(\mu) &= m_\tau(m_\tau)\eta_\tau^{-1}B_e^{-1}\exp(\tilde{c}I_\tau),
\end{aligned} \tag{3.10}$$

where

$$\begin{aligned}
B_u &= \prod\left(\frac{\alpha_i(\mu)}{\alpha_i(m_t)}\right)^{C_i/(2b_i)}, \\
B_d &= \prod\left(\frac{\alpha_i(\mu)}{\alpha_i(m_t)}\right)^{C_i'/(2b_i)}, \\
B_e &= \prod\left(\frac{\alpha_i(\mu)}{\alpha_i(m_t)}\right)^{C_i''/(2b_i)}.
\end{aligned} \tag{3.11}$$

We have

$$I_f(\mu) = \frac{1}{16\pi^2}\int_{\ln m_t}^{\ln\mu} y_f^2(t')\mathrm{d}t'. \tag{3.12}$$

The ratio η_f $(f = u, d, c, s, b, e, \mu, \tau)$ appearing in (3.10) is the QCD–QED rescaling factor for the fermion mass m_f. Integration of (2.5) and (2.6) gives analytic formulas for the running VEVs in the SM, 2HDM, and MSSM:

$$\begin{aligned}
v(\mu) &= v(m_t)\prod\left(\frac{\alpha_i(\mu)}{\alpha_i(m_t)}\right)^{C_i^v/(2b_i)} \times\exp(-3I_t - 3I_b - I_\tau), \\
v_u(\mu) &= v_u(m_t)\prod\left(\frac{\alpha_i(\mu)}{\alpha_i(m_t)}\right)^{C_i^v/(2b_i)}\exp(-3I_t), \\
v_d(\mu) &= v_d(m_t)\prod\left(\frac{\alpha_i(\mu)}{\alpha_i(m_t)}\right)^{C_i^v/(2b_i)}\exp(-3I_b - I_\tau),
\end{aligned} \tag{3.13}$$

where C_i^v has been defined through (2.8) and (2.9). As derived in [14] for the MSSM, the formula for running $\tan\beta(\mu)$ has the same form in the 2HDM at one-loop level:

$$\tan\beta(\mu) = \tan\beta(m_t)\exp\left(-3I_t(\mu) + 3I_b(\mu) + I_\tau(\mu)\right), \tag{3.14}$$

which is obtained by integrating (2.6). The QCD–QED rescaling factors occurring in (3.10) have been determined through the running of $SU(3)_\mathrm{C}\times U(1)_\mathrm{em}$ gauge couplings [26,29,31]

$$\begin{aligned}
\eta_u &= 2.38^{+0.52}_{-0.30}, \\
\eta_s &= \eta_d = 2.36^{+0.53}_{-0.29}, \\
\eta_c &= 2.09^{+0.27}_{-0.19}, \\
\eta_b &= 1.53^{+0.07}_{-0.06}, \\
\eta_e &\approx \eta_\mu \approx \eta_\tau = 1.015.
\end{aligned} \tag{3.15}$$

4 Formulas in MSSM for $M_\mathrm{S} > M_Z$

In the MSSM the natural SUSY scale (M_S) could be very different from the weak scale with $M_\mathrm{S} \approx O$ (TeV), whereas $M_\mathrm{S} \gg 1\,\mathrm{TeV}$ has a gauge hierarchy problem. As our new contribution in MSSM in this paper, compared to [14], we present new analytic formulas for all charged fermion masses for any SUSY scale $M_\mathrm{S} > M_Z$ by running them from $m_t - M_\mathrm{S}$ as in SM and then from $M_\mathrm{S} - \mu$ as in MSSM. We have

$$m_u(\mu) = m_u(1\,\mathrm{GeV})\eta_u^{-1} G_u(\mu), \tag{4.1}$$

$$m_c(\mu) = m_c(m_c)\eta_c^{-1} G_u(\mu) \times \exp\left(\frac{3}{2} I_c(M_\mathrm{S}) + 3\tilde{I}_c(\mu)\right), \tag{4.2}$$

$$m_t(\mu) = m_t(m_t) G_u(\mu) \exp\left(\frac{3}{2} I_t(M_\mathrm{S}) - \frac{3}{2} I_b(M_\mathrm{S}) + 3\tilde{I}_t(\mu) + \tilde{I}_b(\mu)\right), \tag{4.3}$$

$$m_i(\mu) = m_i(1\,\mathrm{GeV})\eta_i^{-1} G_d(\mu), \quad i = d, s, \tag{4.4}$$

$$m_b(\mu) = m_b(m_b)\eta_b^{-1} G_d(\mu) \exp\left(\frac{3}{2} I_b(M_\mathrm{S}) - \frac{3}{2} I_t(M_\mathrm{S}) + 3\tilde{I}_b(\mu) + \tilde{I}_t(\mu)\right), \tag{4.5}$$

$$m_i(\mu) = m_i(1\,\mathrm{GeV})\eta_i^{-1} G_e(\mu), \quad i = e, \mu, \tag{4.6}$$

$$m_\tau(\mu) = m_\tau(m_\tau)\eta_\tau^{-1} G_e(\mu) \times \exp\left(\frac{3}{2} I_\tau(M_\mathrm{S}) + 3\tilde{I}_\tau(\mu)\right), \tag{4.7}$$

where

$$\begin{aligned}
G_u(\mu) &= \left(\frac{\alpha_1(M_\mathrm{S})}{\alpha_1(m_t)}\right)^{\frac{-2}{41}} \left(\frac{\alpha_3(M_\mathrm{S})}{\alpha_3(m_t)}\right)^{\frac{4}{7}} \left(\frac{\alpha_1(\mu)}{\alpha_1(M_\mathrm{S})}\right)^{\frac{-43}{792}} \\
&\quad \times \left(\frac{\alpha_2(\mu)}{\alpha_2(M_\mathrm{S})}\right)^{\frac{-9}{8}} \left(\frac{\alpha_3(\mu)}{\alpha_3(M_\mathrm{S})}\right)^{\frac{8}{9}}, \\
G_d(\mu) &= \left(\frac{\alpha_1(M_\mathrm{S})}{\alpha_1(m_t)}\right)^{\frac{1}{41}} \left(\frac{\alpha_3(M_\mathrm{S})}{\alpha_3(m_t)}\right)^{\frac{4}{7}} \left(\frac{\alpha_1(\mu)}{\alpha_1(M_\mathrm{S})}\right)^{\frac{-19}{792}} \\
&\quad \times \left(\frac{\alpha_2(\mu)}{\alpha_2(M_\mathrm{S})}\right)^{\frac{-9}{8}} \left(\frac{\alpha_3(\mu)}{\alpha_3(M_\mathrm{S})}\right)^{\frac{8}{9}}, \\
G_e(\mu) &= \left(\frac{\alpha_1(M_\mathrm{S})}{\alpha_1(m_t)}\right)^{\frac{-9}{41}} \left(\frac{\alpha_1(\mu)}{\alpha_1(M_\mathrm{S})}\right)^{\frac{-1}{8}} \left(\frac{\alpha_2(\mu)}{\alpha_2(M_\mathrm{S})}\right)^{\frac{-9}{8}}.
\end{aligned} \tag{4.8}$$

Furthermore,

$$\tilde{I}_f(\mu) = \frac{1}{16\pi^2} \int_{\ln M_\mathrm{S}}^{\ln \mu} y_f^2(t')\mathrm{d}t', \tag{4.9}$$

and $I_f(M_\mathrm{S})$ is defined through (3.12) with $\mu = M_\mathrm{S}$. Running of the elements of the CKM matrix in the MSSM leads to modification by the following formulas:

$$|V_{\alpha\beta}(\mu)| = \begin{cases} |V_{\alpha\beta}(m_t)| \exp\left(\frac{3}{2}\left(I_t(M_\mathrm{S}) + I_b(M_\mathrm{S})\right) - \left(\tilde{I}_t(\mu) + \tilde{I}_b(\mu)\right)\right), \\ \qquad \alpha\beta = ub, cb, tb, ts, \\ |V_{\alpha\beta}(m_t)|, \\ \qquad \text{otherwise}. \end{cases} \tag{4.10}$$

The one-loop formulas for $v_u(\mu)$, $v_d(\mu)$ and $\tan\beta(\mu)$ are also modified:

$$\begin{aligned}
v_u(\mu) &= v_u(M_\mathrm{S}) \prod \left(\alpha_i(\mu)\alpha_i(m_t)\right)^{C_i^v/(2b_i)} \times \exp(-3\tilde{I}_t), \\
v_d(\mu) &= v_d(M_\mathrm{S}) \prod \left(\alpha_i(\mu)\alpha_i(m_t)\right)^{C_i^v/(2b_i)} \times \exp(-3\tilde{I}_b - \tilde{I}_\tau), \\
\tan\beta(\mu) &= \tan\beta(M_\mathrm{S}) \times \exp\left(-3\tilde{I}_t(\mu) + 3\tilde{I}_b(\mu) + \tilde{I}_\tau(\mu)\right).
\end{aligned} \tag{4.11}$$

The analytic formulas (4.1)–(4.11) hold good for any value of $m_t < M_\mathrm{S} < \mu$. It may be noted that in the limit of $M_\mathrm{S} \to m_t$, $I_f(M_\mathrm{S}) \to 0$, $\tilde{I}_f(\mu) \to I_f(\mu)$ and the formulas (4.1)–(4.11) reduce to those obtained in [14].

5 Comparison with other formulas

In this section, by comparing with formulas obtained by other authors [20,22,23,25,29,31], we show that our formulas are new and are clearly different. Our numerical computations will be compared with other numerical results in Sect. 6. The basic reasons for the difference of other formulas from ours are that in earlier derivations either the scale dependence of the VEVs has been ignored, or even if it has been included, certain approximations like ignoring all other contributions except those due to the $SU(3)_\mathrm{C}$ gauge and top quark Yukawa couplings have been made. Also, while some other derivations have used a top-down approach containing unknown high-scale masses in the formulas, our formulas contain running masses at low energies determined from experimental data. Our formulas for the SM, 2HDM and MSSM are given in (3.10)–(3.14), (4.1)–(4.11) with the definition of coefficients through (2.8)–(3.5), and they are further explicitly elucidated in Tables 1–3 for the sake of comparison with other formulas.

In the earliest studies of the behaviour of running masses of fermions [22,25] at higher scales, the effect of the scale dependent VEV in the SM has been included to derive analytic formulas for the masses of quarks and leptons of three generations. Using the variable $t' = \ln M/\mu$ and neglecting all one-loop contribution of Yukawa couplings in the RGEs, the following approximate formulas have been derived in the top-down approach, at any lower scale $\mu < M$:

$$v(t') = v(0) \left(\frac{\alpha_1(t')}{\alpha_1(0)}\right)^{\frac{9}{164}} \left(\frac{\alpha_2(t')}{\alpha_2(0)}\right)^{\frac{-27}{76}},$$

Table 1. Comparison of analytic formulas of this analysis with those of [31], where scale dependence of the VEV has been ignored in the non-SUSY standard model. Here η_i ($i = e, \mu, \tau, d, s, b, u, c$) are the QCD–QED rescaling factors given in (3.15)

Reference [31]	This analysis
$m_u(\mu) = m_u(1\,\text{GeV})\eta_u^{-1}A_u^{-1}\exp(3I_t + 3I_b + I_\tau)$	$m_u(\mu) = m_u(1\,\text{GeV})\eta_u^{-1}B_u^{-1}$
$m_c(\mu) = m_c(m_c)\eta_c^{-1}A_u^{-1}\exp(3I_t + 3I_b + I_\tau)$	$m_c(\mu) = m_c(m_c)\eta_c^{-1}B_u^{-1}\exp\left(\frac{3}{2}I_c\right)$
$m_t(\mu) = m_t(m_t)A_u^{-1}\exp\left(\frac{9}{2}I_t + \frac{3}{2}I_b + I_\tau\right)$	$m_t(\mu) = m_t(m_t)B_u^{-1}\exp\left(\frac{3}{2}I_t - \frac{3}{2}I_b\right)$
$m_d(\mu) = m_d(1\,\text{GeV})\eta_d^{-1}A_d^{-1}\exp(3I_t + 3I_b + I_\tau)$	$m_d(\mu) = m_d(1\,\text{GeV})\eta_d^{-1}B_d^{-1}$
$m_s(\mu) = m_s(1\,\text{GeV})\eta_s^{-1}A_d^{-1}\exp(3I_t + 3I_b + I_\tau)$	$m_s(\mu) = m_s(1\,\text{GeV})\eta_s^{-1}B_d^{-1}$
$m_b(\mu) = m_b(m_b)\eta_b^{-1}A_d^{-1}\exp\left(\frac{3}{2}I_t + \frac{9}{2}I_b + I_\tau\right)$	$m_b(\mu) = m_b(m_b)\eta_b^{-1}B_d^{-1}\exp\left(\frac{3}{2}I_b - \frac{3}{2}I_t\right)$
$m_e(\mu) = m_e(1\,\text{GeV})\eta_e^{-1}A_e^{-1}\exp(3I_t + 3I_b + I_\tau)$	$m_e(\mu) = m_e(1\,\text{GeV})\eta_e^{-1}B_e^{-1}$
$m_\mu(\mu) = m_\mu(1\,\text{GeV})\eta_\mu^{-1}A_e^{-1}\exp(3I_t + 3I_b + I_\tau)$	$m_\mu(\mu) = m_\mu(1\,\text{GeV})\eta_\mu^{-1}B_e^{-1}$
$m_\tau(\mu) = m_\tau(m_\tau)\eta_\tau^{-1}A_e^{-1}\exp\left(3I_t + 3I_b + \frac{5}{2}I_\tau\right)$	$m_\tau(\mu) = m_\tau(m_\tau)\eta_\tau^{-1}B_e^{-1}\exp\left(\frac{3}{2}I_\tau\right)$
$v(\mu) = v_0 = 174.11\,\text{GeV}$	$v(\mu) = v(m_t)B_v\exp(-3I_t - 3I_b - I_\tau)$
$A_u = \left(\frac{\alpha_1(\mu)}{\alpha_1(m_t)}\right)^{\frac{17}{164}}\left(\frac{\alpha_2(\mu)}{\alpha_2(m_t)}\right)^{\frac{-27}{76}}\left(\frac{\alpha_3(\mu)}{\alpha_3(m_t)}\right)^{\frac{-4}{7}}$	$B_u = \left(\frac{\alpha_1(\mu)}{\alpha_1(m_t)}\right)^{\frac{2}{41}}\left(\frac{\alpha_3(\mu)}{\alpha_3(m_t)}\right)^{\frac{-4}{7}}$
$A_d = \left(\frac{\alpha_1(\mu)}{\alpha_1(m_t)}\right)^{\frac{5}{164}}\left(\frac{\alpha_2(\mu)}{\alpha_2(m_t)}\right)^{\frac{-27}{76}}\left(\frac{\alpha_3(\mu)}{\alpha_3(m_t)}\right)^{\frac{-4}{7}}$	$B_d = \left(\frac{\alpha_1(\mu)}{\alpha_1(m_t)}\right)^{\frac{-1}{41}}\left(\frac{\alpha_3(\mu)}{\alpha_3(m_t)}\right)^{\frac{-4}{7}}$
$A_e = \left(\frac{\alpha_1(\mu)}{\alpha_1(m_t)}\right)^{\frac{45}{164}}\left(\frac{\alpha_2(\mu)}{\alpha_2(m_t)}\right)^{\frac{-27}{76}}$	$B_e = \left(\frac{\alpha_1(\mu)}{\alpha_1(m_t)}\right)^{\frac{9}{41}}$
	$B_v = \left(\frac{\alpha_1(\mu)}{\alpha_1(m_t)}\right)^{\frac{9}{164}}\left(\frac{\alpha_2(\mu)}{\alpha_2(m_t)}\right)^{\frac{-27}{76}}$

Table 2. Comparison of analytic formulas of this analysis with those of [31], where scale dependence of the VEVs has been ignored in the non-SUSY 2HDM

Reference [31]	This analysis
$m_u(\mu) = m_u(1\,\text{GeV})\eta_u^{-1}A_u^{-1}\exp(3I_t)$	$m_u(\mu) = m_u(1\,\text{GeV})\eta_u^{-1}B_u^{-1}$
$m_c(\mu) = m_c(m_c)\eta_c^{-1}A_u^{-1}\exp(3I_t)$	$m_c(\mu) = m_c(m_c)\eta_c^{-1}B_u^{-1}\exp\left(\frac{3}{2}I_c\right)$
$m_t(\mu) = m_t(m_t)A_u^{-1}\exp\left(\frac{9}{2}I_t + \frac{1}{2}I_b\right)$	$m_t(\mu) = m_t(m_t)B_u^{-1}\exp\left(\frac{3}{2}I_t + \frac{1}{2}I_b\right)$
$m_d(\mu) = m_d(1\,\text{GeV})\eta_d^{-1}A_d^{-1}\exp(3I_b + I_\tau)$	$m_d(\mu) = m_d(1\,\text{GeV})\eta_d^{-1}B_d^{-1}$
$m_s(\mu) = m_s(1\,\text{GeV})\eta_s^{-1}A_d^{-1}\exp(3I_b + I_\tau)$	$m_s(\mu) = m_s(1\,\text{GeV})\eta_s^{-1}B_d^{-1}$
$m_b(\mu) = m_b(m_b)\eta_b^{-1}A_d^{-1}\exp\left(\frac{1}{2}I_t + \frac{9}{2}I_b + I_\tau\right)$	$m_b(\mu) = m_b(m_b)\eta_b^{-1}B_d^{-1}\exp\left(\frac{3}{2}I_b + \frac{1}{2}I_t\right)$
$m_e(\mu) = m_e(1\,\text{GeV})\eta_e^{-1}A_e^{-1}\exp(3I_b + I_\tau)$	$m_e(\mu) = m_e(1\,\text{GeV})\eta_e^{-1}B_e^{-1}$
$m_\mu(\mu) = m_\mu(1\,\text{GeV})\eta_\mu^{-1}A_e^{-1}\exp(3I_b + I_\tau)$	$m_\mu(\mu) = m_\mu(1\,\text{GeV})\eta_\mu^{-1}B_e^{-1}$
$m_\tau(\mu) = m_\tau(m_\tau)\eta_\tau^{-1}A_e^{-1}\exp\left(3I_b + \frac{5}{2}I_\tau\right)$	$m_\tau(\mu) = m_\tau(m_\tau)\eta_\tau^{-1}B_e^{-1}\exp\left(\frac{3}{2}I_\tau\right)$
$v_u(\mu) = v_0\sin\beta(m_t)$	$v_u(\mu) = v_u(m_t)B_v\exp(-3I_t)$
$v_d(\mu) = v_0\cos\beta(m_t)$	$v_d(\mu) = v_d(m_t)B_v\exp(-3I_b - I_\tau)$
$\tan\beta(\mu) = \tan\beta(m_t)$	$\tan\beta(\mu) = \tan\beta(m_t)$ $\times\exp\left(-3I_t(\mu) + 3I_b(\mu) + I_\tau(\mu)\right)$
$A_u = \left(\frac{\alpha_1(\mu)}{\alpha_1(m_t)}\right)^{\frac{17}{168}}\left(\frac{\alpha_2(\mu)}{\alpha_2(m_t)}\right)^{\frac{-3}{8}}\left(\frac{\alpha_3(\mu)}{\alpha_3(m_t)}\right)^{\frac{-4}{7}}$	$B_u = \left(\frac{\alpha_1(\mu)}{\alpha_1(m_t)}\right)^{\frac{1}{21}}\left(\frac{\alpha_3(\mu)}{\alpha_3(m_t)}\right)^{\frac{-4}{7}}$
$A_d = \left(\frac{\alpha_1(\mu)}{\alpha_1(m_t)}\right)^{\frac{5}{168}}\left(\frac{\alpha_2(\mu)}{\alpha_2(m_t)}\right)^{\frac{-3}{8}}\left(\frac{\alpha_3(\mu)}{\alpha_3(m_t)}\right)^{\frac{-4}{7}}$	$B_d = \left(\frac{\alpha_1(\mu)}{\alpha_1(m_t)}\right)^{\frac{-1}{42}}\left(\frac{\alpha_3(\mu)}{\alpha_3(m_t)}\right)^{\frac{-4}{7}}$
$A_e = \left(\frac{\alpha_1(\mu)}{\alpha_1(m_t)}\right)^{\frac{45}{168}}\left(\frac{\alpha_2(\mu)}{\alpha_2(m_t)}\right)^{\frac{-3}{8}}$	$B_e = \left(\frac{\alpha_1(\mu)}{\alpha_1(m_t)}\right)^{\frac{3}{14}}$
	$B_v = \left(\frac{\alpha_1(\mu)}{\alpha_1(m_t)}\right)^{\frac{3}{56}}\left(\frac{\alpha_2(\mu)}{\alpha_2(m_t)}\right)^{\frac{-3}{8}}$

Table 3. Comparison of analytic formulas of this analysis with those of [31], where scale dependence of the VEVs has been ignored in the MSSM

Reference [31]	This analysis
$m_u(\mu) = m_u(1\,\mathrm{GeV})\eta_u^{-1}A_u^{-1}\exp(3I_t)$	$m_u(\mu) = m_u(1\,\mathrm{GeV})\eta_u^{-1}G_u(\mu)$
$m_c(\mu) = m_c(m_c)\eta_c^{-1}A_u^{-1}\exp(3I_t)$	$m_c(\mu) = m_c(m_c)\eta_c^{-1}G_u(\mu)$
	$\times\exp\left(\frac{3}{2}I_c(M_\mathrm{S}) + 3\tilde{I}_c(\mu)\right)$
$m_t(\mu) = m_t(m_t)A_u^{-1}\exp(6I_t + I_b)$	$m_t(\mu) = m_t(m_t)G_u(\mu)$
	$\times\exp\left(\frac{3}{2}I_t(M_\mathrm{S}) - \frac{3}{2}I_b(M_\mathrm{S}) + 3\tilde{I}_t(\mu) + \tilde{I}_b(\mu)\right)$
$m_d(\mu) = m_d(1\,\mathrm{GeV})\eta_d^{-1}A_d^{-1}\exp(3I_b + I_\tau)$	$m_d(\mu) = m_d(1\,\mathrm{GeV})\eta_d^{-1}G_d(\mu)$
$m_s(\mu) = m_s(1\,\mathrm{GeV})\eta_s^{-1}A_d^{-1}\exp(3I_b + I_\tau)$	$m_s(\mu) = m_s(1\,\mathrm{GeV})\eta_s^{-1}G_d(\mu)$
$m_b(\mu) = m_b(m_b)\eta_b^{-1}A_d^{-1}\exp(I_t + 6I_b + I_\tau)$	$m_b(\mu) = m_b(m_b)\eta_b^{-1}G_d(\mu)$
	$\times\exp\left(\frac{3}{2}I_b(M_\mathrm{S}) - \frac{3}{2}I_t(M_\mathrm{S}) + 3\tilde{I}_b(\mu) + \tilde{I}_t(\mu)\right)$
$m_e(\mu) = m_e(1\,\mathrm{GeV})\eta_e^{-1}A_e^{-1}\exp(3I_b + I_\tau)$	$m_e(\mu) = m_e(1\,\mathrm{GeV})\eta_e^{-1}G_e(\mu)$
$m_\mu(\mu) = m_\mu(1\,\mathrm{GeV})\eta_\mu^{-1}A_e^{-1}\exp(3I_b + I_\tau)$	$m_\mu(\mu) = m_\mu(1\,\mathrm{GeV})\eta_\mu^{-1}G_e(\mu)$
$m_\tau(\mu) = m_\tau(m_\tau)\eta_\tau^{-1}A_e^{-1}\exp(3I_b + 4I_\tau)$	$m_\tau(\mu) = m_\tau(m_\tau)\eta_\tau^{-1}G_e(\mu)$
	$\times\exp\left(\frac{3}{2}I_\tau(M_\mathrm{S}) + 3\tilde{I}_\tau(\mu)\right)$
$v_u(\mu) = v_0\sin\beta(m_t)$	$v_u(\mu) = v_u(M_\mathrm{S})G_v\exp(-3\tilde{I}_t)$
$v_d(\mu) = v_0\cos\beta(m_t)$	$v_d(\mu) = v_d(M_\mathrm{S})G_v\exp(-3\tilde{I}_b - \tilde{I}_\tau)$
$\tan\beta(\mu) = \tan\beta(m_t)$	$\tan\beta(\mu) = \tan\beta(M_\mathrm{S})$
	$\times\exp\left(-3\tilde{I}_t(\mu) + 3\tilde{I}_b(\mu) + \tilde{I}_\tau(\mu)\right)$
$A_u = \left(\frac{\alpha_1(\mu)}{\alpha_1(m_t)}\right)^{\frac{13}{198}}\left(\frac{\alpha_2(\mu)}{\alpha_2(m_t)}\right)^{\frac{-3}{2}}\left(\frac{\alpha_3(\mu)}{\alpha_3(m_t)}\right)^{\frac{-8}{9}}$	$G_u(\mu) = \left(\frac{\alpha_1(M_\mathrm{S})}{\alpha_1(m_t)}\right)^{\frac{-2}{41}}\left(\frac{\alpha_3(M_\mathrm{S})}{\alpha_3(m_t)}\right)^{\frac{4}{7}}$
	$\times\left(\frac{\alpha_1(\mu)}{\alpha_1(M_\mathrm{S})}\right)^{\frac{-43}{792}}\left(\frac{\alpha_2(\mu)}{\alpha_2(M_\mathrm{S})}\right)^{\frac{-9}{8}}\left(\frac{\alpha_3(\mu)}{\alpha_3(M_\mathrm{S})}\right)^{\frac{8}{9}}$
$A_d = \left(\frac{\alpha_1(\mu)}{\alpha_1(m_t)}\right)^{\frac{7}{198}}\left(\frac{\alpha_2(\mu)}{\alpha_2(m_t)}\right)^{\frac{-3}{2}}\left(\frac{\alpha_3(\mu)}{\alpha_3(m_t)}\right)^{\frac{-8}{9}}$	$G_d(\mu) = \left(\frac{\alpha_1(M_\mathrm{S})}{\alpha_1(m_t)}\right)^{\frac{1}{41}}\left(\frac{\alpha_3(M_\mathrm{S})}{\alpha_3(m_t)}\right)^{\frac{4}{7}}$
	$\times\left(\frac{\alpha_1(\mu)}{\alpha_1(M_\mathrm{S})}\right)^{\frac{-19}{792}}\left(\frac{\alpha_2(\mu)}{\alpha_2(M_\mathrm{S})}\right)^{\frac{-9}{8}}\left(\frac{\alpha_3(\mu)}{\alpha_3(M_\mathrm{S})}\right)^{\frac{8}{9}}$
$A_e = \left(\frac{\alpha_1(\mu)}{\alpha_1(m_t)}\right)^{\frac{3}{22}}\left(\frac{\alpha_2(\mu)}{\alpha_2(m_t)}\right)^{\frac{-3}{2}}$	$G_e(\mu) = \left(\frac{\alpha_1(M_\mathrm{S})}{\alpha_1(m_t)}\right)^{\frac{-9}{41}}\left(\frac{\alpha_1(\mu)}{\alpha_1(M_\mathrm{S})}\right)^{\frac{-1}{8}}\left(\frac{\alpha_2(\mu)}{\alpha_2(M_\mathrm{S})}\right)^{\frac{-9}{8}}$
	$G_v = \left(\frac{\alpha_1(\mu)}{\alpha_1(m_t)}\right)^{\frac{1}{88}}\left(\frac{\alpha_2(\mu)}{\alpha_2(m_t)}\right)^{\frac{3}{8}}$

$$
\begin{aligned}
m_i(t') &= m_i(0)\left(\frac{\alpha_1(t')}{\alpha_1(0)}\right)^{\frac{-9}{41}}, \quad i = e,\mu,\tau,\\
m_j(t') &= m_j(0)\left(\frac{\alpha_1(t')}{\alpha_1(0)}\right)^{\frac{-2}{41}}\left(\frac{\alpha_3(t')}{\alpha_3(0)}\right)^{\frac{4}{7}}, \quad j = u,c,t,\\
m_k(t') &= m_k(0)\left(\frac{\alpha_1(t')}{\alpha_1(0)}\right)^{\frac{1}{41}}\left(\frac{\alpha_3(t')}{\alpha_3(0)}\right)^{\frac{4}{7}}, \quad k = d,s,b.
\end{aligned} \tag{5.1}
$$

Comparing with (5.1), our formulas in the SM summarised in Table 1 and given in (3.10)–(3.14) contain the additional contributions of relevant Yukawa couplings of the third generation. Whereas the formulas in (5.1) contain the unknown high-scale masses $m_i(0)$ ($i = e,\mu,\tau,u,c,t,d,s,b$), our formulas and the VEV $v(0)$ formulas contain the VEV and experimentally measurable masses at low energies through the QCD–QED rescaling factors as given in (3.15).

While unravelling the RG fixed point behaviour of top quark Yukawa coupling in the SM, Pendleton and Ross [23] did include the scale dependence in the VEV of the SM Higgs scalar to derive the following formulas for the running masses of the up and down quark masses using the variable $t = (1/2)\ln\left(\mu^2/\mu_0^2\right)$ and the dominance of the QCD gauge and top quark Yukawa couplings over all other couplings:

$$
m_i(t) = m_i(t_0)\left(\frac{\alpha_3(t)}{\alpha_3(t_0)}\right)^{\frac{4}{7}}, \quad i = u,c, \tag{5.2}
$$

$$
\begin{aligned}
m_j(t) &= m_j(t_0)\left(\frac{\alpha_3(t)}{\alpha_3(t_0)}\right)^{\frac{4}{7}}\\
&\quad\times\left[1 + \frac{9y_t^2(t)}{8\pi\alpha_3(t_0)}\left\{\left(\frac{\alpha_3(t)}{\alpha_3(t_0)}\right)^{\frac{1}{7}}\right\} - 1\right]^{|\lambda_{tD}|^2/6},\\
j &= d,s,b,
\end{aligned} \tag{5.3}
$$

where

$$
\lambda_{tD} \approx \left(1 - \frac{y_b^2}{y_t^2}\right). \tag{5.4}
$$

In contrast to (5.2)–(5.3) our formulas do not contain unknown high-scale parameters, but our predictions at higher scales are made in terms of experimentally measured parameters as inputs. In addition, our analytic formulas represent the one-loop radiative corrections due to all gauge couplings of the SM and all the third generation Yukawa couplings. Besides, we have also obtained new formulas for all quarks and charged leptons.

Using the $\overline{\mathrm{MS}}$ scheme in the SM, Balzeleit et al [20] have utilised the effect of the scale dependent VEV and the dominance of QCD gauge and top quark Yukawa couplings to obtain the following analytic formulas for the running masses of quarks at higher scales in terms of those at lower scales ($\mu_0 = 1\,\mathrm{GeV}$),

$$\begin{aligned} m_i(\mu) &= m_i(\mu_0) G_3(\mu), \quad i = u, d, c, s, \\ m_b(\mu) &= m_b(\mu_0) \left[1 + \frac{9 m_t^2(\mu_0)}{2\pi\alpha_3(\mu_0) v^2(\mu_0)} \right. \\ &\quad \left. \times \left\{ \left(\frac{\alpha_3(\mu)}{\alpha_3(\mu_0)} \right)^{\frac{1}{7}} - 1 \right\} \right]^{\frac{1}{6}} G_3(\mu), \\ m_t(\mu) &= m_t(\mu_0) \left[1 + \frac{9 m_t^2(\mu_0)}{2\pi\alpha_3(\mu_0) v^2(\mu_0)} \right. \\ &\quad \left. \times \left\{ \left(\frac{\alpha_3(\mu)}{\alpha_3(\mu_0)} \right)^{\frac{1}{7}} - 1 \right\} \right]^{\frac{-1}{6}} G_3(\mu), \\ G_3(\mu) &= \left(\frac{\alpha_3(\mu)}{\alpha_3(\mu_0)} \right)^{\frac{4}{7}} . \end{aligned} \tag{5.5}$$

As can be verified from (3.10) and Table 1, our formulas in the corresponding cases contain additional contributions due to $SU(2) \times U(1)$ gauge couplings and all third generation Yukawa couplings. Further, we have new formulas for charged lepton masses.

As noted in Sects. 2 and 3, the RGEs for the scale dependent VEVs in the SM and MSSM along with those for other couplings have been obtained by Arason et al. [16–18] but no analytic formulas have been derived by them.

Using a top-down approach, the third generation effects on fermion mass predictions have been examined earlier in MSSM [29], where the scale dependence of the VEVs and the corresponding RGEs have been ignored. In the bottom-up approach, using the same renormalisation scheme, analytic formulas have been explicitly specified in [31] also ignoring the scale dependence of the VEVs. In Tables 1–3 we compare our analytic formulas with those of [31] in SM, 2HDM and MSSM. Whereas the top quark Yukawa coupling integral defined through (3.12) has been predicted to affect the running of $m_u(\mu)$ and $m_c(\mu)$ [29, 31], our formulas predict no such effect. Similarly, whereas the b quark and the τ lepton Yukawa coupling integrals have been predicted to affect the running charged lepton masses $m_e(\mu)$ and $m_\mu(\mu)$ [29,31], our formulas predict no such contributions. In particular, our formulas predict that in all the three effective gauge theories, SM, 2HDM, or MSSM, the third generation Yukawa couplings do not affect the running masses of the first two generations in the small mixing limit and at one-loop level. This is in clear contrast to the results of [29,31] where the influence of the third generation effects have been emphasised on the running masses of the first two generations. Another important and new feature of our formulas is that even for the running masses of the third generation, the Yukawa coupling integrals occur in the exponents with different coefficients when compared with earlier analytic formulas [29,31].

The dependence on the gauge couplings can also be noted to be quite different in our analytic formulas. Whereas earlier derivations [29,31] predicted the occurrence of the exponents $C_i^u/2b_i$, $C_i^d/2b_i$, and $C_i^e/2b_i$ on the R.H.S. of (3.11), our formulas predict the corresponding exponents to be $C_i/2b_i$, $C_i'/2b_i$, and $C_i''/2b_i$, respectively. This has led to the gauge coupling factors, B_u, B_d, and B_e, to be different from the corresponding factors A_u, A_d, and A_e in the earlier formulas [29,31] as shown in Tables 1–3 for each gauge theory. Thus our formulas at one-loop level predict a substantially new functional dependence on gauge and Yukawa couplings for the running masses in SM, 2HDM, and MSSM when compared with those obtained by ignoring scale dependent VEVs [29,31].

When scale dependence in the corresponding VEVs is ignored, there are no RGEs for $v_u(\mu)$ and $v_d(\mu)$, in 2HDM nor in MSSM. This assumption gives $\tan\beta(\mu) = \tan\beta(m_t)$ for all higher scales $\mu > m_t$ [29,31]. But inclusion of the scale dependence of the VEVs through their RGEs in (2.6) naturally leads to the new analytic formula for $\tan\beta(\mu)$ given in (3.14) and (4.11) and is explained through comparison in Tables 2–3.

Also our formulas for the case of MSSM are the same as those obtained in [14] when the SUSY scale is assumed to be $M_\mathrm{S} = m_t$. But for any SUSY scale $M_\mathrm{S} > m_t$, our formulas given in (4.1)–(4.11) are new and have been derived for the first time. In the limit $M_\mathrm{S} = m_t$, the formulas (4.1)–(4.11) reduce to those in (3.10) and [14].

In the next section, while making numerical predictions of the running masses at higher scales at the two-loop level in SM and MSSM and at one-loop level in 2HDM we have made comparative studies with earlier numerical estimations wherever they exist.

6 Numerical predictions at higher scales

The analytic formulas given in the previous section predict masses and CKM matrix elements up to the one-loop level at higher scales. We have also numerically estimated the effect of scale dependent VEVs on predictions of the running masses at two-loop level. We solve the RGEs for the Yukawa matrices and VEVs including two-loop contributions in SM and MSSM [16–19,26–28] numerically and obtain the mass matrices at higher scales from the corresponding products of the two. For this purpose, the elements of the CKM matrix at higher scales have been obtained by running them through the one-loop RGEs given by (3.8) with appropriate values of the coefficient c given in (2.10)–(2.12) [29,30]. In 2HDM we carry out all numerical estimations at one-loop level. We use the following inputs for the running masses (m_i), SM gauge couplings (α_1, α_2, α_3), electromagnetic fine structure constant (α),

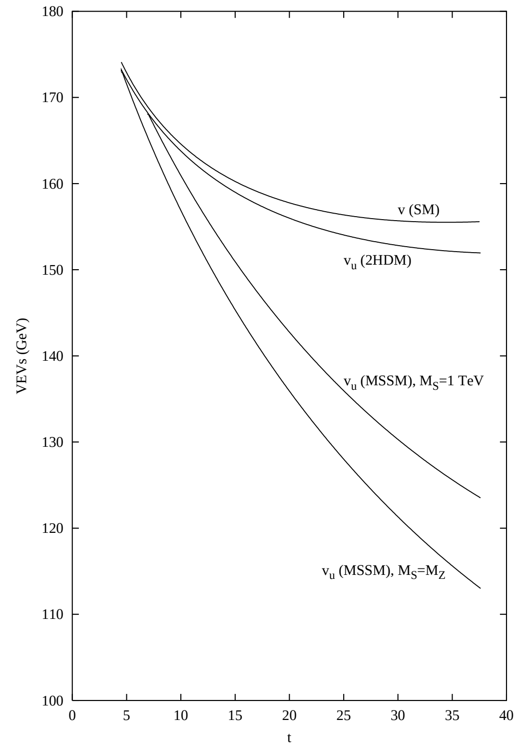

Fig. 1. Variation of running VEVs in the SM, 2HDM and MSSM as a function of $\mu(t = \ln \mu)$ showing a substantial deviation from the scale independence assumption

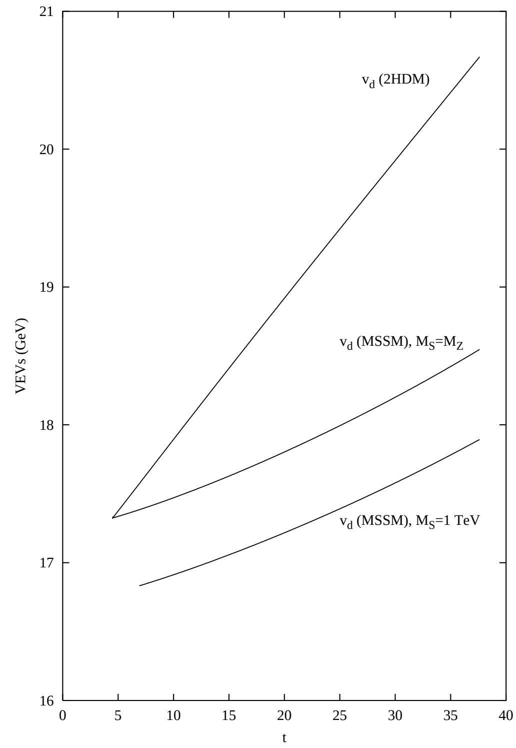

Fig. 2. Variation of running VEVs at higher scales in MSSM and 2HDM as a function of $\mu(t = \ln \mu)$ showing a substantial deviation from the scale independence assumption

electroweak mixing angle and the CKM matrix (V) at $\mu = M_Z$ which have been obtained from the experimental data [26,29,31,33]:

$$\begin{aligned} m_u &= 2.33^{+0.42}_{-0.45}\,\mathrm{MeV}, \quad m_c = 677^{+56}_{-61}\,\mathrm{MeV}, \\ m_t &= 181 \pm 13\,\mathrm{GeV}, \quad m_d = 4.69^{+0.60}_{-0.66}\,\mathrm{MeV}, \\ m_s &= 93.4^{+11.8}_{-13.0}\,\mathrm{MeV}, \quad m_b = 3.00 \pm 0.11\,\mathrm{GeV}, \\ m_e &= 0.48684727 \pm 0.00000014\,\mathrm{MeV}, \\ m_\mu &= 102.75138 \pm 0.00033\,\mathrm{MeV}, \\ m_\tau &= 1.74669^{+0.00030}_{-0.00027}\,\mathrm{GeV}. \end{aligned} \tag{6.1}$$

$$\begin{aligned} \alpha_1(M_Z) &= 0.016829 \pm 0.000017, \\ \alpha_2(M_Z) &= 0.033493^{+0.000042}_{-0.000038}, \\ \alpha_3(M_Z) &= 0.118 \pm 0.003, \\ \alpha_{\mathrm{em}}^{-1} &= 128.896 \pm 0.09, \\ \sin^2\theta_{\mathrm{W}} &= 0.23165 \pm 0.000024. \end{aligned} \tag{6.2}$$

Also, we have

$$V(M_Z) = \begin{pmatrix} 0.9757 & 0.2205 & 0.0030\mathrm{e}^{-\mathrm{i}\delta} \\ -0.2203 - 0.0001\mathrm{e}^{\mathrm{i}\delta} & 0.9747 & 0.0373 \\ 0.0082 - 0.0029\mathrm{e}^{\mathrm{i}\delta} & -0.0364 - 0.0007\mathrm{e}^{\mathrm{i}\delta} & 0.9993 \end{pmatrix}. \tag{6.3}$$

For the sake of convenience we have used $\delta = \pi/2$ as in [33]. The choice of the same input quantities enables us to compare our results on mass predictions with those obtained with the scale independence assumption on the VEVs in SM and MSSM [33]. We neglect mixings among charged leptons and use the diagonal basis for up quarks.

The variations of the VEVs as a function of μ are shown in Figs. 1 and 2 for the SM, 2HDM, and MSSM where the initial value of $\tan\beta(M_Z) = 10$ has been used for the latter two cases. In these and certain other figures we have used the variable $t = \ln\mu$ along the X-axis where μ is in units of GeV. It is quite clear that in the SM as well as the other cases the running effects of the VEVs contribute to very significant departures from the assumed scale independent values [29,31–33]. Thus, the predicted running masses are to be different in all three cases. Since $v_u(\mu)$ decreases and $v_d(\mu)$ increases with increasing μ, the up quark masses are expected to have decreasing effects, whereas the down quark and charged lepton masses are expected to have increasing effects at higher scales in MSSM and 2HDM. But in the SM all the masses are expected to have decreasing effects due to the decreasing value of $v(\mu)$. In fact, these features are clearly exhibited in all numerical values of the mass predictions carried out in this investigation. It is to be noted that almost all fermion masses, except the top quark, the b quark and the τ lepton near the perturbative limits, decrease at higher scales due to the decrease in the corresponding Yukawa couplings. But the effect of running VEVs contribute to additional decreasing or increasing factors in the respective cases.

The predictions of all the charged fermion masses as a function of $t = \ln\mu$ are shown in Fig. 3 with $M_{\mathrm{S}} = M_Z$

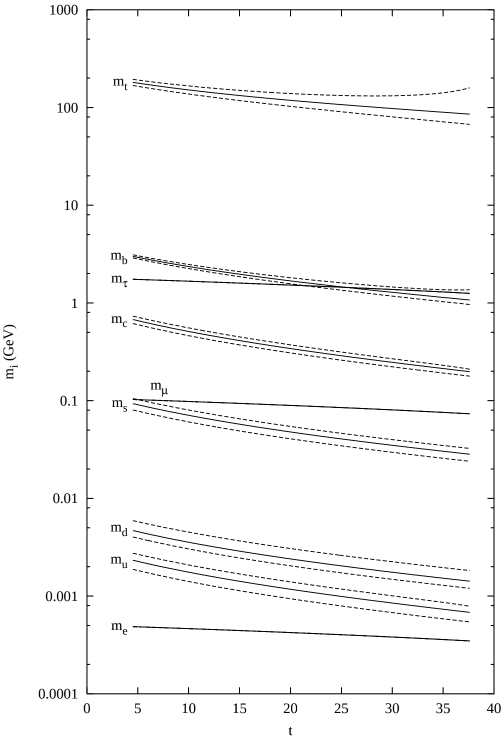

Fig. 3. Predictions of running masses at higher scales as a function of μ ($t = ln\mu$) in MSSM with SUSY scale $M_S = M_Z$ using the input parameters given in (6.1)–(6.3) and $\tan\beta(M_S) = 10$. The dashed lines are due to uncertainties in the input parameters

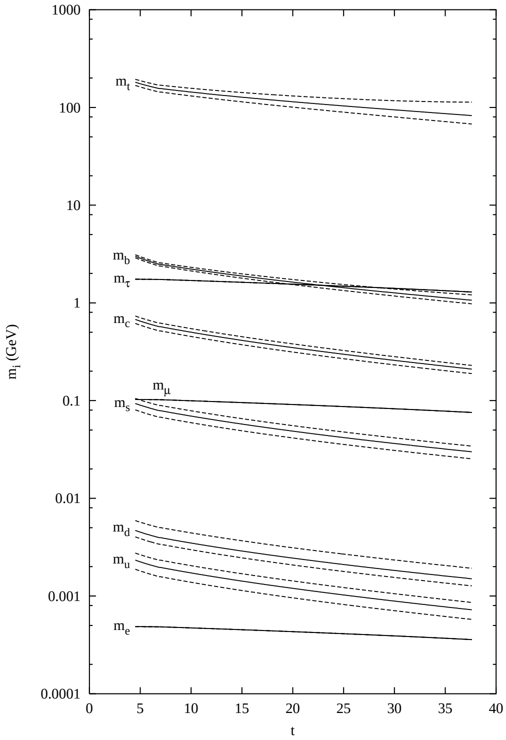

Fig. 4. Same as Fig. 3 but with $M_S = 1\,\mathrm{TeV}$

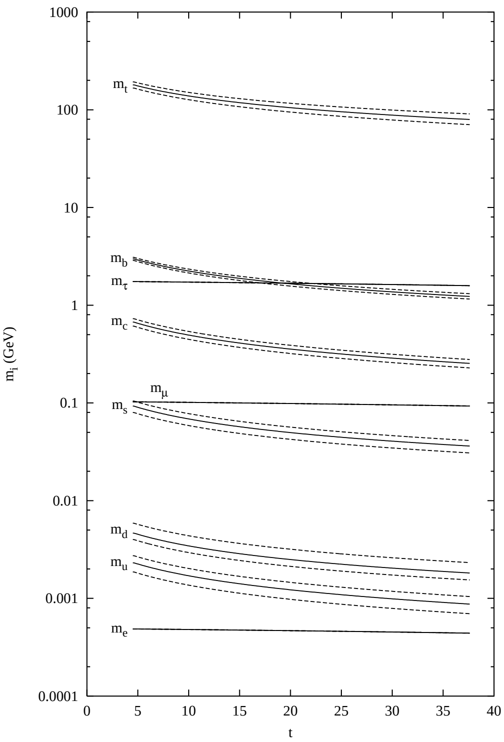

Fig. 5. Predictions of running masses at higher scales in the 2HDM using the input parameters given in (6.1)–(6.3) and $\tan\beta(M_Z) = 10$

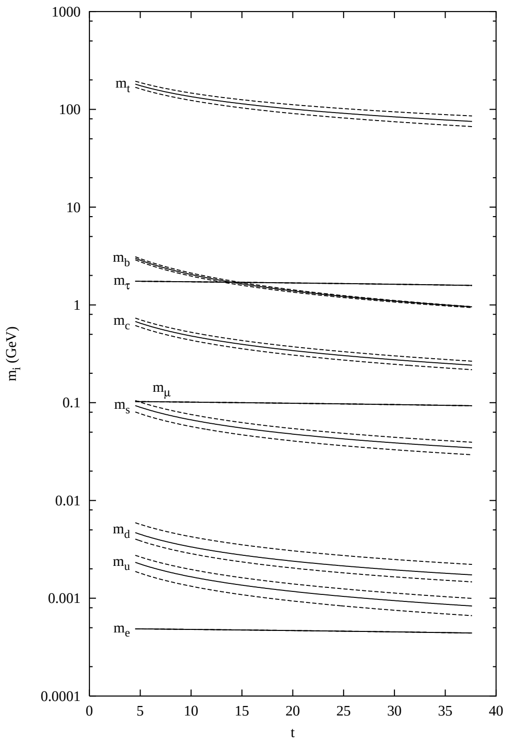

Fig. 6. Predictions of running masses at higher scales in SM with the input parameters given in (6.1)–(6.3) and $M_M = 250\,\mathrm{GeV}$

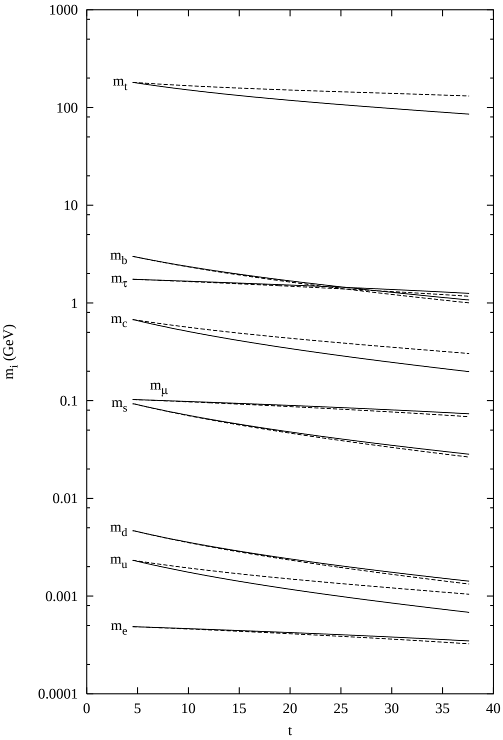

Fig. 7. Comparison of running mass predictions in the MSSM (solid lines) with those obtained from scale independence assumptions (dashed lines) on the VEVs. The SUSY scale has been taken to be M_Z

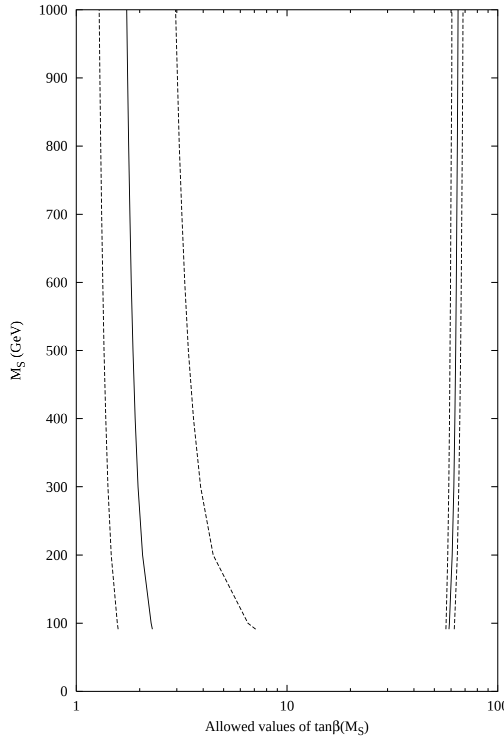

Fig. 8. Perturbatively allowed region for $\tan\beta(M_\mathrm{S})$ as a function of SUSY scale M_S. The lower (upper) limits are due to top quark (b quark) Yukawa coupling. The dashed lines are due to uncertainties in the respective input masses

and in Fig. 4 with $M_\mathrm{S} = 1\,\mathrm{TeV}$ in the case of MSSM using $\tan\beta(M_Z) = 10$. The corresponding predictions in 2HDM and SM are shown in Figs. 5 and 6. Our numerical estimations agree very closely with the corresponding mass predictions at high scales in 2HDM by Cvetic et al. [21] for $m_s(\mu)$, $m_b(\mu)$, $m_c(\mu)$ and $m_t(\mu)$. But in 2HDM too we have additional numerical estimations, both in the quark and in the lepton sector. In Fig. 7 we display the comparison of the mass predictions as functions of $t = \ln\mu$ with and without running VEVs in MSSM assuming $M_\mathrm{S} = M_Z$ and $\tan\beta = 10$. Although the differences in the two types of predictions are clearly distinguishable, they are quite prominent in the up quark sectors. While the new contributions are seen to be significant for the down quarks and charged leptons at higher scales with $\mu \geq 10^7\,\mathrm{GeV}$, in the case of up quarks the contributions are found to be important starting from $\mu = O$ (TeV). As compared to the scale independence assumption [33], our predictions are clearly smaller for the up quarks and larger for the down quarks and charged leptons as indicated by solid-line curves in Fig. 7. With the input values for m_t and m_b in (6.1), the lowest allowed value of $\tan\beta(M_\mathrm{S})$ is determined by observing the perturbative limit for the top quark Yukawa coupling at the GUT scale, $y_t^2(M_\mathrm{GUT})/4\pi \leq 1.0$ and the highest allowed value of $\tan\beta(M_\mathrm{S})$ is determined from the corresponding limit on the b quark Yukawa coupling.

MSSM

$$M_\mathrm{S} = M_Z : 2.3^{+4.8}_{-0.6} \leq \tan\beta(M_\mathrm{S}) \leq 58.7^{+3.4}_{-2.0}, \quad (6.4)$$

$$M_\mathrm{S} = 1\,\mathrm{TeV} : 1.7^{+1.3}_{-0.4} \leq \tan\beta(M_\mathrm{S}) \leq 64.8^{+3.6}_{-4.3}. \quad (6.5)$$

The allowed region for $\tan\beta(M_\mathrm{S})$ as a function of M_S in MSSM is shown in Fig. 8 where the solid (dashed) lines are due to the central values (uncertainties) in the inputs of m_t and m_b. It is clear that the allowed region for $\tan\beta$ increases, although slowly, with increasing M_S. In the 2HDM the allowed region for $\tan\beta$ is found to be substantially larger.

2HDM

$$1.2^{+0.3}_{-0.2} \leq \tan\beta(M_Z) \leq 68.9 \pm 2.7. \quad (6.6)$$

We have noted that in all three effective theories, the difference between the one- and two-loop estimates of the running masses at the highest scale (M_U) varies between 1–5%, the lowest discrepancy being for the leptons and the highest being for the top quark. But in MSSM and 2HDM this discrepancy increases to 10–12% for the b and the top quarks as the respective perturbative limits are approached.

The running VEVs in MSSM and 2HDM lead to a variation of $\tan\beta(\mu)$ as a function of μ over its initial value at M_Z. This is shown in Fig. 9 for different input values where the dashed (solid) line represents the case for 2HDM (MSSM). In both theories $\tan\beta(\mu)$ decreases (increases) from its initial value when the latter crosses a critical point. This critical value is $\tan\beta(M_Z) \approx 56$ (52) in MSSM (2HDM). In Fig. 10 we present $\tan\beta(M_U)$ at the GUT scale as a function of $\tan\beta(M_Z)$ for both theories. We observe a steep rise in the curves as the respective

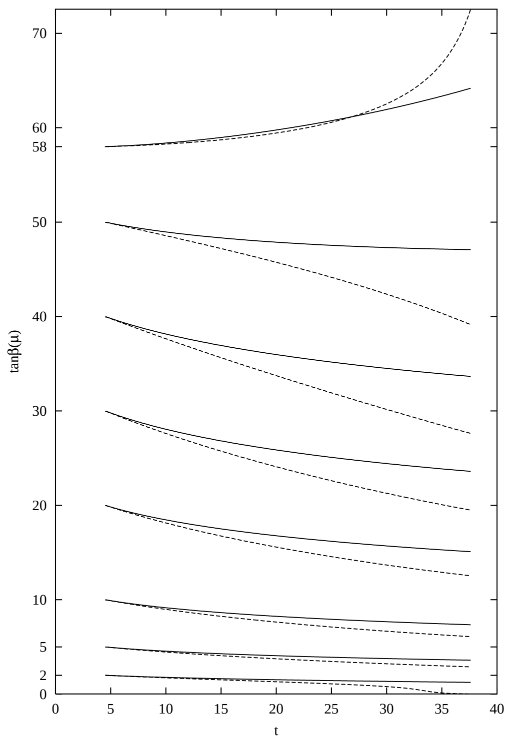

Fig. 9. Variation of $\tan\beta(\mu)$ as a function of μ ($t = \ln\mu$) for different input values of $\tan\beta(M_Z)$ in MSSM (solid lines) and 2HDM (dashed lines). The inputs for different curves starting from the bottom most line is $\tan\beta(M_Z)$ =2, 5, 10, 20, 30, 40, 50 and 58. In the MSSM the SUSY scale has been taken as M_Z

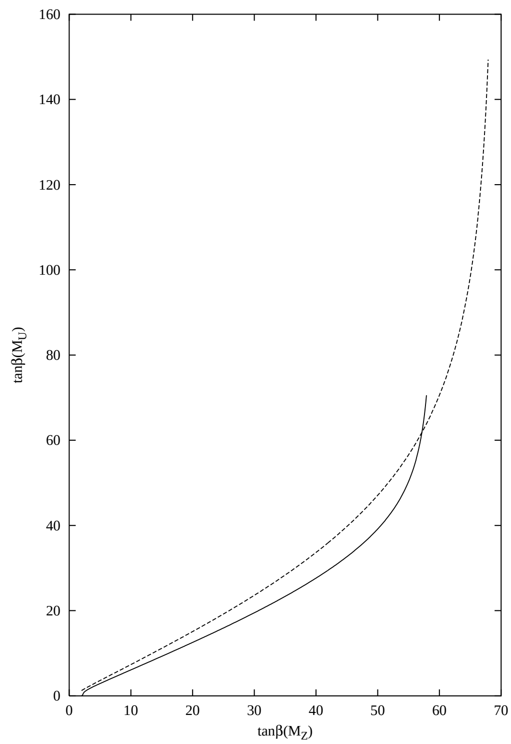

Fig. 10. Predictions of $\tan\beta(M_U)$ as function of $\tan\beta(M_Z)$ in MSSM (solid line) and 2HDM (dashed line). In MSSM the SUSY scale has been taken as M_Z

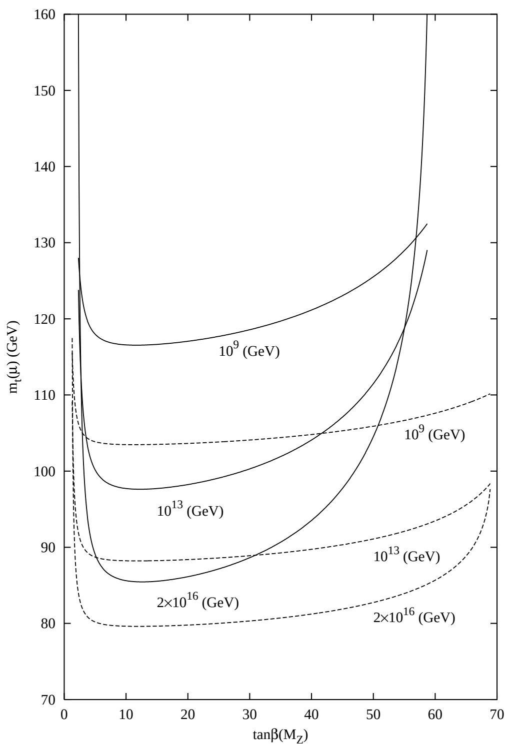

Fig. 11. Prediction of $m_t(\mu)$ at higher scales, $\mu = 10^9$ GeV, 10^{13} GeV and 2×10^{16} GeV as a function of $\tan\beta(M_Z)$ in MSSM with $M_\mathrm{S} = M_Z$ (solid lines) and 2HDM (dashed lines)

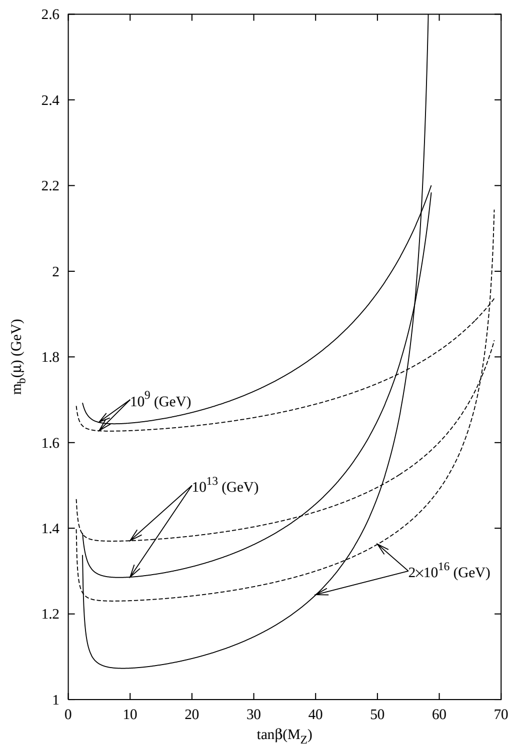

Fig. 12. Predicton of $m_b(\mu)$ at higher scales, $\mu = 10^9$ GeV, 10^{13} GeV and 2×10^{16} GeV as a function of $\tan\beta(M_Z)$ in MSSM with $M_\mathrm{S} = M_Z$ (solid lines) and 2HDM (dashed lines)

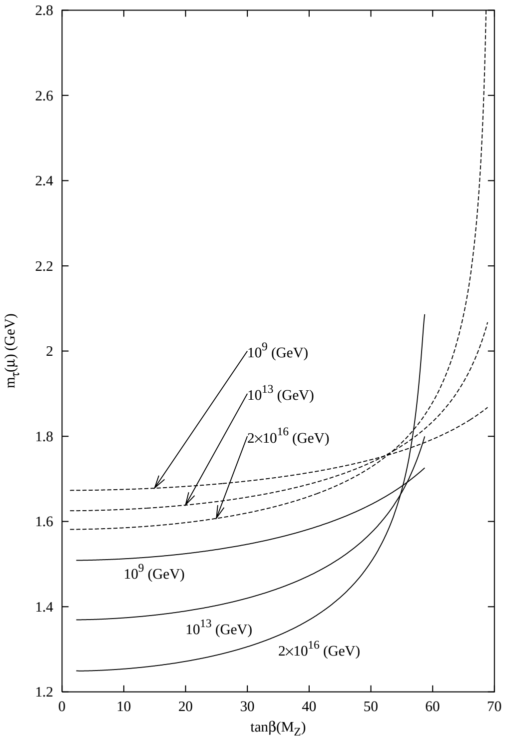

Fig. 13. Predicton of $m_\tau(\mu)$ as at higher scales, $\mu = 10^9$ GeV, 10^{13} GeV and $2{\times}10^{16}$ GeV as a function of $\tan\beta(M_Z)$ in MSSM with $M_{\rm S} = M_Z$ (solid lined) and 2HDM (dashed lines)

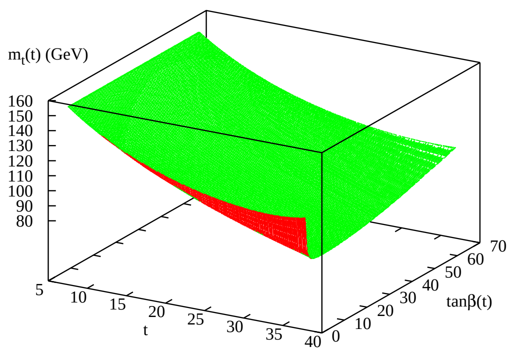

Fig. 14. Prediction of top quark mass $m_t(\mu)$ at higher scales ($\mu > M_Z$) as a function of μ ($t = \ln\mu$) and $\tan\beta(\mu)$ in MSSM with $M_{\rm S} = 1$ TeV. The values of $\tan\beta(\mu)$ at very μ has been obtained through solutions of the corresponding RGE using $\tan\beta(M_Z) = 2$–58 as inputs

perturbative limits are approached in the large $\tan\beta(M_Z)$ region.

Using the central values of $m_t(M_Z)$, $m_b(M_Z)$ and $m_\tau(M_Z)$ from (6.1), we have studied the variation of $m_t(\mu)$, $m_b(\mu)$ and $m_\tau(\mu)$ for the different values of $\mu = 10^9$ GeV, 10^{13} GeV and 2×10^{16} GeV, each as a function of various low-energy input values of $\tan\beta(M_Z)$ in MSSM and 2HDM. These results are presented in Figs. 11–13 for the 2HDM (dashed lines) and for the MSSM (solid lines) with $M_{\rm S} = M_Z$. It is clear that the perturbatively allowed range of $\tan\beta$ decreases with increasing μ both for MSSM and 2HDM.

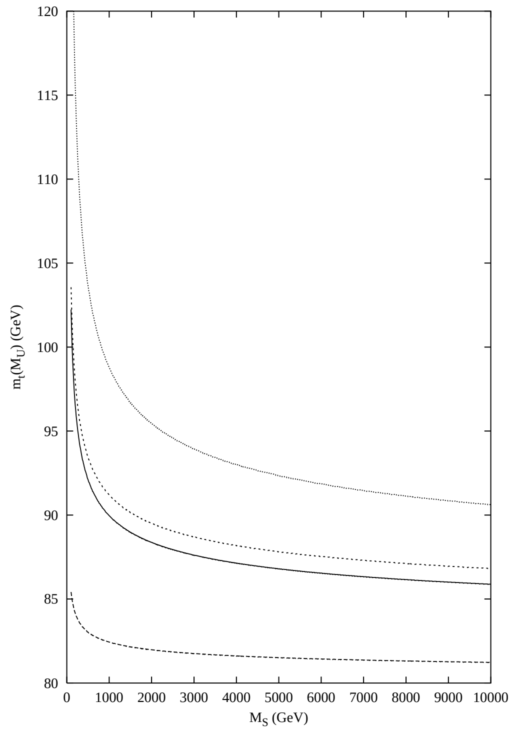

Fig. 15. Variation of top quark mass prediction at the GUT scale as a function of the SUSY scale $M_{\rm S} = M_Z - 10^4$ GeV and various values of $\tan\beta(M_{\rm S})$=3 (solid line), 10 (large-dashed line), 50 (small-dashed line), 55 (dotted line)

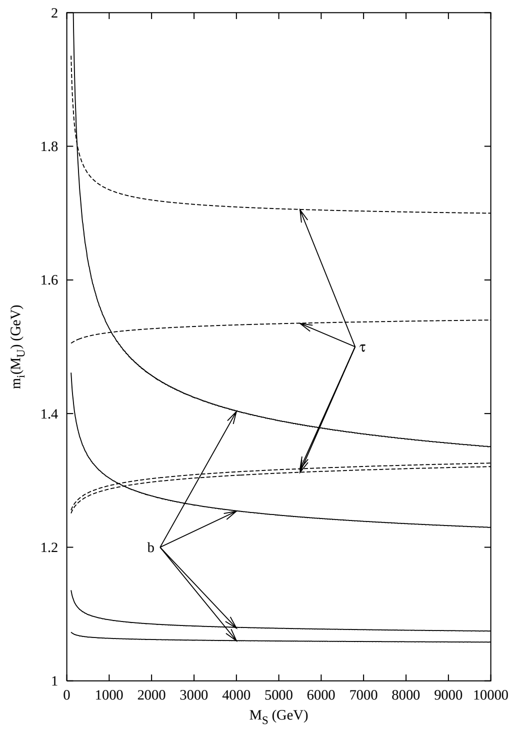

Fig. 16. Same as Fig. 15 but for b quark and τ lepton mass predictions. The input values of $\tan\beta(M_{\rm S})$ for four different curves in each case are $\tan\beta(M_{\rm S}) = 3$, 10, 50, and 55 in the increasing order of masses

Table 4. Running mass and VEV predictions at higher scales in the non-SUSY standard model for the input values of the Higgs mass $M_{\mathrm{M}} = 250\,\mathrm{GeV}$ and other parameters given in (6.1)–(6.3)

	$\mu = 10^9$ (GeV)	$\mu = 10^{13}$ (GeV)	$\mu = 2 \times 10^{16}$ (GeV)
m_u (MeV)	$1.1537^{+0.2233}_{-0.2331}$	$0.9472^{+0.1849}_{-0.1923}$	$0.8351^{+0.1636}_{-0.1700}$
m_c (MeV)	$335.2184^{+31.8261}_{-33.5603}$	$275.2419^{+26.5286}_{-27.8710}$	$242.6476^{+23.5536}_{-24.7026}$
m_t (GeV)	$99.1359^{+10.7438}_{-9.8347}$	$83.9249^{+10.2622}_{-9.0281}$	$75.4348^{+9.9647}_{-8.5401}$
m_d (MeV)	$2.3558^{+0.6513}_{-0.3538}$	$1.9529^{+0.5433}_{-0.2953}$	$1.7372^{+0.4846}_{-0.2636}$
m_s (MeV)	$46.9155^{+6.5228}_{-6.9737}$	$38.8929^{+5.4652}_{-5.8228}$	$34.5971^{+4.8857}_{-5.1971}$
m_b (GeV)	$1.3639^{+0.0328}_{-0.0398}$	$1.0971^{+0.0143}_{-0.0248}$	$0.9574^{+0.0037}_{-0.0169}$
m_e (MeV)	$0.4665^{+0.0001}_{-0.0001}$	$0.4533^{+0.0001}_{-0.0001}$	$0.4413^{+0.0001}_{-0.0001}$
m_μ (MeV)	$98.4648^{+0.0049}_{-0.0050}$	$95.6834^{+0.0078}_{-0.0084}$	$93.1431^{+0.0136}_{-0.0101}$
m_τ (GeV)	$1.6738^{+0.0004}_{-0.0003}$	$1.6265^{+0.0005}_{-0.0004}$	$1.5834^{+0.0001}_{-0.0005}$
v (GeV)	$157.5206^{-7.1815}_{+6.0558}$	$155.7062^{-10.6592}_{-8.5945}$	$155.6196^{-13.6336}_{+10.4664}$

Table 5. Predictions of running masses, VEVs and $\tan\beta$ at higher scales $\mu = 10^9$ GeV, 10^{13} GeV and 2×10^{16} GeV in MSSM with SUSY scale $M_{\mathrm{S}} = 1\,\mathrm{TeV}$, using two-loop RG equations

$\tan\beta(M_{\mathrm{S}}) = 10$	$\mu = 10^9$ (GeV)	$\mu = 10^{13}$ (GeV)	$\mu = 2 \times 10^{16}$ (GeV)
m_u (MeV)	$1.1618^{+0.2226}_{-0.2345}$	$0.8882^{+0.1694}_{-0.1794}$	$0.7238^{+0.1365}_{-0.1467}$
m_c (MeV)	$339.4064^{+31.2929}_{-33.4804}$	$258.0945^{+23.8287}_{-25.8339}$	$210.3273^{+19.0036}_{-21.2264}$
m_t (GeV)	$112.3144^{+17.0392}_{-13.7215}$	$94.3698^{+22.5577}_{-14.4831}$	$82.4333^{+30.2676}_{-14.7686}$
m_d (MeV)	$2.3842^{+0.6582}_{-0.3574}$	$1.8290^{+0.5111}_{-0.2779}$	$1.5036^{+0.4235}_{-0.2304}$
m_s (MeV)	$47.4812^{+6.5845}_{-7.0454}$	$36.4261^{+5.1588}_{-5.4807}$	$29.9454^{+4.3001}_{-4.5444}$
m_b (GeV)	$1.5920^{+0.1038}_{-0.0915}$	$1.2637^{+0.1189}_{-0.0893}$	$1.0636^{+0.1414}_{-0.0865}$
m_e (MeV)	$0.4290^{+0.0001}_{-0.0001}$	$0.3911^{+0.0002}_{-0.0002}$	$0.3585^{+0.0003}_{-0.0003}$
m_μ (MeV)	$90.5439^{+0.0169}_{-0.0173}$	$82.5539^{+0.0346}_{-0.0330}$	$75.6715^{+0.0578}_{-0.0501}$
m_τ (GeV)	$1.5429^{+0.0006}_{-0.0006}$	$1.4085^{+0.0009}_{-0.0008}$	$1.2922^{+0.0013}_{-0.0012}$
$\tan\beta$	$8.2314^{-0.5046}_{+0.3807}$	$7.4350^{-0.9752}_{+0.6302}$	$6.9280^{-1.5156}_{+0.8234}$
v_u (GeV)	$141.7765^{-9.7365}_{+7.6253}$	$130.5455^{-18.0431}_{+12.1155}$	$123.8177^{-27.8954}_{+15.7651}$
v_d (GeV)	$17.2237^{-0.1352}_{+0.1241}$	$17.5581^{-0.1426}_{+0.1302}$	$17.8718^{-0.1492}_{+0.1354}$
$\tan\beta(M_{\mathrm{S}}) = 55$	$\mu = 10^9$ (GeV)	$\mu = 10^{13}$ (GeV)	$\mu = 2 \times 10^{16}$ (GeV)
m_u (MeV)	$1.1687^{+0.2225}_{-0.2346}$	$0.8889^{+0.1675}_{-0.1795}$	$0.7244^{+0.1219}_{-0.1466}$
m_c (MeV)	$339.5917^{+31.2621}_{-33.5026}$	$258.2929^{+23.3295}_{-25.8144}$	$210.5049^{+15.1077}_{-21.1538}$
m_t (GeV)	$118.6588^{+19.9035}_{-15.4790}$	$104.2363^{+32.7015}_{-18.2028}$	$95.1486^{+69.2836}_{-20.659}$
m_d (MeV)	$2.3774^{+0.6542}_{-0.3553}$	$1.8219^{+0.5054}_{-0.2755}$	$1.4967^{+0.4157}_{0.2278}$
m_s (MeV)	$47.3523^{+6.5303}_{-7.0069}$	$36.2891^{+5.0777}_{-5.4340}$	$29.8135^{+4.1795}_{-4.4967}$
m_b (GeV)	$1.8297^{+0.1667}_{-0.1376}$	$1.5768^{+0.2640}_{-0.1685}$	$1.4167^{+0.4803}_{-0.1944}$
m_e (MeV)	$0.4276^{-0.0003}_{+0.0001}$	$0.3893^{-0.0005}_{+0.0002}$	$0.3565^{-0.001}_{+0.0002}$
m_μ (MeV)	$90.2779^{-0.0508}_{+0.0318}$	$82.2064^{-0.1024}_{+0.0468}$	$75.2938^{-0.1912}_{+0.0515}$
m_τ (GeV)	$1.6867^{+0.0056}_{-0.005}$	$1.6574^{+0.0188}_{-0.0148}$	$1.6292^{+0.0443}_{-0.0294}$
$\tan\beta$	$53.6122^{-2.3644}_{+1.5356}$	$52.7633^{-6.3597}_{+2.9538}$	$52.0738^{-16.5475}_{+4.3757}$
v_u (GeV)	$141.2095^{-10.6285}_{+8.1355}$	$127.4742^{-22.6973}_{+13.8538}$	$117.7947^{-46.7214}_{+19.2752}$
v_d (GeV)	$2.6339^{-0.0859}_{+0.0741}$	$2.4159^{-0.158}_{+0.1206}$	$2.2620^{-0.2615}_{+0.1661}$

Table 6. Predictions of running masses, VEVs and $\tan\beta$ in 2HDM at higher scales using one-loop RG equations

$\tan\beta(M_\mathrm{S}) = 10$	$\mu = 10^9$ (GeV)	$\mu = 10^{13}$ (GeV)	$\mu = 2\times10^{16}$ (GeV)
m_u (MeV)	$1.2021^{+0.2309}_{-0.2417}$	$0.9908^{+0.1919}_{-0.2002}$	$0.8749^{+0.1701}_{-0.1772}$
m_c (MeV)	$349.2805^{+32.6824}_{-34.5798}$	$287.8975^{+27.3606}_{-28.8305}$	$254.2131^{+24.3398}_{-25.5998}$
m_t (GeV)	$103.5011^{+11.3400}_{-10.2307}$	$88.2332^{+11.1753}_{-9.5397}$	$79.6373^{+11.1974}_{-9.127}$
m_d (MeV)	$2.4547^{+0.6748}_{-0.366}$	$2.0430^{+0.5650}_{-0.3069}$	$1.8204^{+0.505}_{-0.2743}$
m_s (MeV)	$48.8852^{+6.7278}_{-7.2144}$	$40.6860^{+5.6602}_{-6.0484}$	$36.2544^{+5.0700}_{-5.4083}$
m_b (GeV)	$1.6281^{+0.0910}_{-0.0854}$	$1.3709^{+0.0854}_{-0.0775}$	$1.2309^{+0.0826}_{-0.0730}$
m_e (MeV)	$0.4662^{+0.0001}_{-0.0001}$	$0.4529^{+0.0001}_{-0.0001}$	$0.4407^{+0.0001}_{-0.0001}$
m_μ (MeV)	$98.4132^{+0.0050}_{-0.0051}$	$95.5970^{+0.0086}_{-0.0086}$	$93.0197^{+0.0122}_{-0.0122}$
m_τ (GeV)	$1.6752^{+0.0004}_{-0.0004}$	$1.6283^{+0.0004}_{-0.0004}$	$1.5851^{+0.0005}_{-0.0005}$
$\tan\beta$	$8.1956^{-0.3894}_{+0.3255}$	$7.6757^{-0.5649}_{+0.4496}$	$7.3543^{-0.6975}_{+0.5348}$
v_u (GeV)	$155.6481^{-7.4729}_{+6.2622}$	$152.8315^{-11.3442}_{+9.0595}$	$151.9551^{-14.5219}_{+11.1741}$
v_d (GeV)	$18.9914^{-0.0098}_{+0.0095}$	$19.9110^{-0.0137}_{+0.0131}$	$20.6620^{-0.0167}_{+0.0157}$
$\tan\beta(M_\mathrm{S}) = 55$	$\mu = 10^9$ (GeV)	$\mu = 10^{13}$ (GeV)	$\mu = 2\times10^{16}$ (GeV)
m_u (MeV)	$1.2021^{+0.2309}_{-0.2417}$	$0.9908^{+0.1919}_{-0.2002}$	$0.8749^{+0.1701}_{-0.1772}$
m_c (MeV)	$349.2889^{+32.6905}_{-34.5782}$	$287.9066^{+27.3646}_{-28.8338}$	$254.2223^{+24.3441}_{-25.6031}$
m_t (GeV)	$106.6700^{+12.2719}_{-10.9231}$	$92.1000^{+12.6648}_{-10.5174}$	$83.9317^{+13.2279}_{-10.3226}$
m_d (MeV)	$2.4547^{+0.6748}_{-0.3660}$	$2.0430^{+0.5650}_{-0.3069}$	$1.8204^{+0.5050}_{-0.2743}$
m_s (MeV)	$48.8888^{+6.7295}_{-7.2159}$	$40.6898^{+5.6622}_{-6.0498}$	$36.2584^{+5.0720}_{-5.4099}$
m_b (GeV)	$1.7719^{+0.1203}_{-0.1092}$	$1.5392^{+0.1272}_{-0.1092}$	$1.4128^{+0.1353}_{-0.1162}$
m_e (MeV)	$0.4662^{+0.0001}_{-0.0001}$	$0.4529^{+0.0001}_{-0.0001}$	$0.4407^{+0.0001}_{0.0001}$
m_μ (MeV)	$98.4302^{+0.0054}_{-0.0055}$	$95.6235^{+0.0097}_{-0.0094}$	$93.0536^{+0.0146}_{-0.0136}$
m_τ (GeV)	$1.7659^{+0.0028}_{-0.0025}$	$1.7775^{+0.0073}_{-0.0060}$	$1.7851^{+0.0136}_{-0.0107}$
$\tan\beta$	$54.9963^{-1.5534}_{+1.1589}$	$55.7094^{-2.4787}_{+1.6588}$	$56.5831^{-3.2730}_{+1.9895}$
v_u (GeV)	$155.7178^{-7.8644}_{+6.5265}$	$152.0846^{-12.3141}_{+9.6439}$	$150.4478^{-16.1914}_{+12.0879}$
v_d (GeV)	$2.8314^{-0.0649}_{+0.0578}$	$2.7299^{-0.1042}_{+0.0892}$	$2.6588^{-0.1404}_{+0.1161}$

We have examined the simultaneous variation of $m_t(\mu)$ as a function of μ and $\tan\beta(\mu)$ which is displayed in the three dimensional plot of Fig. 14 for the input value of $m_t(M_Z) = 181$ GeV and $\tan\beta(M_\mathrm{S}) = 2$–$58$, where $M_\mathrm{S} = 1$ TeV. Using the top quark mass at $\mu = M_Z$, we have calculated $m_t(\mu)$ and $\tan\beta(\mu)$ at every μ between $M_\mathrm{S} - M_U$ for the input value of $\tan\beta(M_\mathrm{S}) = 2$–$59$. The results are displayed in the three dimensional plot. The variations of the running mass predictions at the GUT scale ($M_U = 2\times10^{16}$ GeV) as a function of the SUSY scale ($M_\mathrm{S} = M_Z - 10^4$ GeV) are shown in Figs. 15 and 16 for the third generation fermions using various input values of $\tan\beta(M_\mathrm{S})$. We find that the top quark mass at the GUT scale at first decreases sharply in the smaller and larger $\tan\beta$ regions as M_S increases and then remains almost constant for $M_\mathrm{S} = 3\times10^3$–$10^4$ GeV. Similarly the predicted b and τ masses decrease with increasing M_S although the fall-off is slower in the case of τ in the large $\tan\beta$ region.

Numerical values of predictions of the running masses are presented in Table 4 for the SM at the three different scales $\mu = 10^9$ GeV, 10^{13} GeV and 2×10^{16} GeV. The two-loop contributions to the RG evolution of the Yukawa couplings depends, although very weakly, upon the Higgs quartic couplings λ, which is related to the Higgs mass (M_H) and VEV (v), $\lambda = M_\mathrm{H}^2/(4v^2)$. We have used the two-loop RGEs for $\lambda(\mu)$ for the SM [17] and evaluated the running masses and VEVs of Table 4 for the input value of the Higgs mass $M_\mathrm{H} = 250$ GeV. Changing the Higgs mass in the allowed range of $M_\mathrm{M} = 220$–260 GeV [33] does not change the results of Table 4 significantly. The uncertainties in the quantities are due to those in the running masses at $\mu = M_Z$. The mass matrices for $M_b(\mu)$ and $M_u(\mu)$ are modified by the factors $v(\mu)/v(M_Z)$ where $v(M_Z) \approx 174$ GeV and the CKM matrices at higher scales remain the same as in [33]. The computed values of masses are found to be less when compared to those obtained with the scale independence assumption [33]. This is clearly understood as the running VEV in the SM decreases with increasing μ. For example, in the SM at $\mu = 2\times10^{16}$ GeV our predictions are $(m_u, m_c, m_t) = (0.83\,\mathrm{MeV}, 242.6\,\mathrm{MeV}, 75.4\,\mathrm{GeV})$ as compared to [33] $(m_u, m_c, m_t) = (0.94\,\mathrm{MeV}, 272\,\mathrm{MeV}, 84\,\mathrm{GeV})$ where the running effect on the VEV has been ignored. In Tables 5 and 6 numerical values of masses, VEVs and $\tan\beta$ are given at the same three scales for the MSSM and 2HDM with $\tan\beta(M_Z) = 10$ and 55 in each of the two theories. As emphasised in this paper our high-scale estimations predict quite significantly different values for the running masses, especially in the up quark sector. Although the CKM matrices at high scales remain the same as under scale independence assumptions on the VEVs, the up quark mass matrices are modified by the factor $v_u(\mu)/v_u(M_S)$, but the down quark and charged lepton mass matrices are modified by the factor $v_d(\mu)/v_d(M_S)$. In the MSSM with $M_S = M_Z$ and $\tan\beta(M_Z) = 10$, including the running effect of the VEVs, the GUT scale predictions are $(m_u, m_c, m_t) = (0.70\,\text{MeV}, 200\,\text{MeV}, 73.5\,\text{GeV})$ as compared to [33] $(m_u, m_c, m_t) = (1.04\,\text{MeV}, 302\,\text{MeV}, 129\,\text{GeV})$. But by increasing the SUSY scale to $M_S = 1\,\text{TeV}$ and in the large $\tan\beta$ region, we find a substantial decrease in the predicted values of the top quark mass at the GUT scale, leading to $(m_u, m_c, m_t) = (0.72\,\text{MeV}, 210\,\text{MeV}, 95.1\,\text{GeV})$. This is understood by noting that $\tan\beta \approx 55$ is closer to the perturbative limit for which the top quark Yukawa coupling is larger. Similarly from Table 5 we note a nearly 20% increase in the $m_b(M_U)$ with the increase of $\tan\beta$ from 10–55. Similar effects are also noted in 2HDM as can be seen from Table 6, where m_t decreases by nearly 7% as $\tan\beta$ increases from 10–55. For a larger effect the increase has to be larger in $\tan\beta$ since the perturbative limit in this case is closer to $\tan\beta \approx 69$ as compared to the MSSM case, where the limiting value is $\tan\beta \approx 59$.

7 Summary and conclusion

In the frequently exploited renormalisation scheme in gauge theories, the Yukawa couplings and VEVs in the SM, 2HDM and MSSM run separately [16–31,33]. The effect of scale dependence of the VEVs has been ignored while deriving analytic formulas [29,31] and obtaining numerical predictions at higher scales for the running masses of fermions [33], but it has been considered in [16–25] and it has more appropriately been taken into account more recently [14] for MSSM. In this paper, we have derived new analytic formulas in the SM and 2HDM and generalised the formulas of [14] for any supersymmetry scale ($M_S > M_Z$). The analytic formulas are given in (3.10)–(3.14), summarised explicitly and compared with formulas derived by other authors in Tables 1–3. The new formulas exhibit a substantially different functional dependence on gauge and Yukawa couplings in all the three effective theories. In particular, the running masses of the first two generations are found to be independent of the Yukawa couplings of the third generations in the small mixing limit. Numerical predictions at two-loop level show that all the running masses in the SM and only the up quark masses in the MSSM and 2HDM decrease at high scales when compared with the predictions taking scale independent VEVs. But in the case of MSSM and 2HDM, the down quark and the charged lepton masses increase over the corresponding predictions obtained with scale independent assumptions on the VEVs. Compared to the MSSM the perturbatively allowed region of $\tan\beta$ is larger in 2HDM. In MSSM the allowed region shows a slow increase with the SUSY scale. We have also made predictions of the running masses at the GUT scale as a function of supersymmetry scale, exhibiting new behaviours. We suggest that these new analytic formulas and improved estimates on the running masses and $\tan\beta$ at high scales be used as inputs to test models proposing unified explanations of the quark and lepton masses.

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